CMPUT 204 — Problem Set 1 Partial Solution

(Partially provided by *Xiaohu Lu*)

Topics covered in Part I are mathematical background (Appendix A, pages 1058–1069; Appendix C, pages 1094–1126), mathematical induction and loop invariant, running time (function) comparison, and insertion sort; in Part II are algorithm analysis using insertion sort as an example and algorithm design using merge sort as an example, growth of functions, and asymptotic notations.

It is highly recommended that you read pages **1–61**, **1058–1069**, and **1094–1126** very carefully and do **all** the exercises. The following are some of them that you are **REQUIRED** to practice on.

Quiz questions are mostly based on this list, with some minor modifications necessary. Consult your instructor and TAs if you have any problem with this list.

Part I

1. The following theorem is attributed to Nicomachus (c. 100 A.D.):

$$\begin{aligned} 1^3 &= 1, \\ 2^3 &= 3+5, \\ 3^3 &= 7+9+11, \\ 4^3 &= 13+15+17+19, \\ &\cdots \text{ et cetera } \cdots \end{aligned}$$

   (a) Give the expression of n^3 for general n.
   Hint:

      i. total number of the right hand side is n.

      ii. all numbers of the right hand side are sequential odd numbers.

      iii. the first number of the right hand side follows a certain pattern: there are $(\sum_{i=1}^{n-1} i)$ odd natural numbers ahead of it, therefore it is $(\sum_{i=1}^{n-1} i+1)_{th}$ odd natural number. It can be expressed as $2(\sum_{i=1}^{n-1} i+1)-1$, which can be simplified as n^2-n+1

      iv.

$$\begin{aligned} n^3 = & \sum_{i=1}^{n}(n^2-n+1)+2*(i-1)) \\ = & \sum_{i=1}^{n}(n^2-n+2*i-1) \end{aligned}$$

   (b) Prove your expression.
   Hint:
   Either you can directly derive from the right hand side of the expression to get n^3, or you can prove it by mathematical induction.

      i. base case: $n=1$

      ii. induction step:
      assume it holds for $n=k$, i.e, $k^3 = \sum_{i=1}^{k}(k^2-k+2*i-1)$, prove it also holds for $n=k+1$.

2. The following proof by induction seems correct, but for some reason the equation for $n=6$ gives $5/6$ on the left-hand side, and $4/3$ on the right-hand side. Can you find a mistake?

   THEOREM: $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\cdots+\frac{1}{(n-1)\times n}=\frac{3}{2}-\frac{1}{n}$.

   Proof: We use induction on n. For $n=1$, $\frac{1}{1\times 2}=\frac{3}{2}-\frac{1}{n}$; and, assuming the theorem is true for n,

$$\begin{aligned} & \frac{1}{1\times 2}+\cdots+\frac{1}{(n-1)\times n}+\frac{1}{n\times(n+1)} \\ = & \frac{3}{2}-\frac{1}{n}+\frac{1}{n(n+1)} \\ = & \frac{3}{2}-\frac{1}{n}+(\frac{1}{n}-\frac{1}{n+1}) \\ = & \frac{3}{2}-\frac{1}{n+1}. \end{aligned}$$

   Hint:
   pay attention to the left hand side of the THEOREM, see what should be the base case if we do mathematical induction proof on it. Don't always assume base case be $n=1$.

3. How can $\log_2 n$ be computed on a pocket calculator that computes only natural logarithms of the form $\ln x$? Justify your answer. Start from the basic definition of logarithms.
   Hint:
   $log_2 n = \frac{\ln n}{\ln 2}$

4. Five distinct elements are randomly chosen from integers between 1 and 20, and stored in a list $L[1]$, ..., $L[5]$. Using linear search we want to determine if an integer x (also chosen randomly from integers between 1 and 20) belongs to the list L.

   (a) What is the number of comparisons required on the average?
   Hint:
   Check for for the probability that you need exactly 1 comparison, which is $\frac{1}{20}$ (why?). What about 2 comparisons? The answer is 4.5 comparisons on average.

   (b) Give a similar analysis as in the first part if L has n elements and all numbers are selected from integers between 1 and m.
   Hint:
   $\frac{2mn-n^2+n}{2m}$.

5. P13, Prob 1-1.

   Using functions $\lg n$, n, n^2, 2^n, and times 1 second, 1 day, 1 century.

6. P21, Ex 2.1-1.

7. P21, Ex 2.1-3.
   Hint:

   (a) pseudo-code:

   ```
   i ← 1
   while i ≤ length[A] and A[i] ≠ v
       do i ← i + 1
   if i > length[A]
       i ← NIL
   return i
   ```

   (b) loop invariant:
   At the start of each iteration of the **while** loop, the subarray $\mathbf{A}[1...i-1]$ does not contain v.
   Prove it on three properties: initialization, maintenance, and termination.

8. P21, Ex 2.1-4.
   Hint:
   consider $\mathbf{A}$ and $\mathbf{B}$ are two n-element arrays with binary elements, adding them up is simply to iterate both arrays from tail to head and do binary addition on each pair of bits then fill the result into the corresponding position in $\mathbf{C}$.

   Make sure you take care of the carriage.