Lecture 30: Monday March 31, 2003

today

- divide and conquer: matrix multiplication
- NP completeness

announcements

new problem: matrix multiplication

- definition of $C = A_{n \times m} * B_{m \times n}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & \ldots & a_{1m} \\ a_{21} & a_{22} & \ldots & \ldots & a_{2m} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & \ldots & a_{nm} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ b_{m1} & b_{m2} & \ldots & b_{mn} \end{bmatrix}$$

$$C = \begin{bmatrix} c_{11} & c_{12} & \ldots & c_{1n} \\ c_{21} & c_{22} & \ldots & c_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ c_{n1} & c_{n2} & \ldots & c_{nn} \end{bmatrix} \qquad c_{jk} = \begin{bmatrix} a_{j1} & a_{j2} & \ldots & a_{jm} \end{bmatrix} \begin{bmatrix} b_{1k} \\ b_{2k} \\ \ldots \\ \ldots \\ b_{mk} \end{bmatrix}$$

divide and conquer matrix multiplication

- partition A, B into 4 submatrices
  - assume $n = m$
  - $A = \begin{bmatrix} A_{11} & | & A_{12} \\ -- & & -- \\ A_{21} & | & A_{22} \end{bmatrix}$ and $B = \begin{bmatrix} B_{11} & | & B_{12} \\ -- & & -- \\ B_{21} & | & B_{22} \end{bmatrix}$
  - $C' \leftarrow \begin{bmatrix} A_{11} * B_{11} + A_{12} * B_{21} & | & A_{11} * B_{12} + A_{12} * B_{22} \\ ----------- & & ----------- \\ A_{21} * B_{11} + A_{22} * B_{21} & | & A_{21} * B_{12} + A_{22} * B_{22} \end{bmatrix}$
  - $C' = C$ (check: $c'_{jk} = c_{jk}$ for each j, k) ☺

analysis

- $A(n), M(n)$: total number of add'ns, mult'ns

$$M(n) = \begin{cases} 8M(n/2) & \text{if } n \geq 2 \\ 1 & \text{if } n = 1 \end{cases}$$

$$A(n) = \begin{cases} 8A(n/2) + n^2 & \text{if } n \geq 2 \\ 0 & \text{if } n = 1 \end{cases}$$

- $M(n)$: $a = 8$, $b = 2$, $c = 0$
  $a > b^c$ so $M(n) \in \Theta(n^{\lg_2 8}) = \Theta(n^3)$
- $A(n)$: $a = 8$, $b = 2$, $c = 2$
  $a > b^c$ so $A(n) \in \Theta(n^{\lg_2 8}) = \Theta(n^3)$
- conclusion: same as ordinary matrix mult'n ☹

Strassen's divide/conquer matrix mult'n [CLRS 28.2]

$$\begin{array}{rcl}
X_1 & \leftarrow & (A_{11}+A_{22})*(B_{11}+B_{22}) \\
X_2 & \leftarrow & (A_{21}+A_{22})*B_{11} \\
X_3 & \leftarrow & A_{11}*(B_{12}-B_{22}) \\
X_4 & \leftarrow & A_{22}*(B_{21}-B_{11}) \\
X_5 & \leftarrow & (A_{11}+A_{12})*B_{22} \\
X_6 & \leftarrow & (A_{21}-A_{11})*(B_{11}+B_{12}) \\
X_7 & \leftarrow & (A_{12}-A_{22})*(B_{21}+B_{22})
\end{array}$$

$$C^* \leftarrow \left[\begin{array}{c|c} X_1+X_4-X_5+X_7 & X_3+X_5 \\ \hline X_2+X_4 & X_1+X_3-X_2+X_6 \end{array}\right]$$

- correctness: $C^* = C$ (exercise)
- analysis:

$$M(n) = \begin{cases} 7M(n/2) & \text{if } n \geq 2 \\ 1 & \text{if } n = 1 \end{cases}$$

$$A(n) = \begin{cases} 7A(n/2) + \frac{18}{4}n^2 & \text{if } n \geq 2 \\ 0 & \text{if } n = 1 \end{cases}$$

  - exercise: give r.r. for $T(n) = A(n) + M(n)$
  - $M(n)$: $a = 7$, $b = 2$, $c = 0$
    $a > b^c$ so $M(n) \in \Theta(n^{\lg_2 7}) = \Theta(n^{2.81\ldots})$
  - $A(n)$: $a = 7$, $b = 2$, $c = 2$
    $a > b^c$ so $A(n) \in \Theta(n^{\lg_2 7}) = \Theta(n^{2.81\ldots})$
  - conclusion: better than ordinary ☺

divide/conquer matrix multiplication: conclusions

- ordinary $\Theta(n^3)$
- Strassen $\Theta(n^{2.81\ldots})$
- in practice
  - for small n — ordinary (not d/c) beats Strassen
  - for larger n — Strassen beats ordinary
- current best algorithm — approx $\Theta(n^{2.376})$
- best lower bound — (number of array elements) $\Omega(n^2)$
- how hard is matrix multiplication?
  - not sure (open problem) — between these two bounds

NP-Completeness [CLRS Ch 34]

- simple path: no repeated vertex
- shortest $x - y$ path? an alg'm in $O(n^2)$
- longest $x - y$ path? no known alg'm in $O(n^k)$ for any k
- Eulerian tour: cycle which includes every graph edge exactly once
- Eulerian graph: has Eulerian tour
- Hamiltonian cycle: includes every graph vertex exactly once
- Hamiltonian graph: has Hamiltonian cycle
- graph Eulerian? an alg'm in $O(n^2)$
- graph Hamiltonian? no known alg'm in $O(n^k)$ for any k
- the class of problems P
  - decision problem (output: yes or no)
  - some alg'm solves problem in poly'l time
  - which of above problems are in P?
- the class of problems NP
  - decision problem (output: yes or no)
  - for every instance with answer yes, there is a proof that the answer is yes which can be verified in polynomial time
  - which of above problems are in P?