# Lecture 9: Friday January 24, 2003

## today

- Chapter 4: recurrences
  - iterated substitution
  - simple version of master theorem

## recall: solving recurrence relations

- iterated substitution (done)
- recursion trees (done)
- master theorem (now)
- what form do D&C r.r.'s usually have?
- consider the following D&C procedure:

```
proc yada(n)
    ...   ...   ...   ...   ...   ...   ...
    ... yada(n/b)  ...  yada(n/b) ...
    ...   ...   ...   ...   ...   ...   ...
    ...   ...   return   ...
endYada
```

- for the call `yada(n)` assume:
  - run time (excluding recursive calls) is $n^c$
  - there are a total of $a$ calls to `yada(n/b)`
- recurrence relation for total time $T(n)$:

$$T(n) = \begin{cases} aT(\frac{n}{b}) + n^c & \text{if } \; n \geq b \\ \text{bounded} & \text{if } \; n < b \end{cases}$$

- closed form solution?
  - repeated substitution
  - simplifying assumption: $n = b^k$

$$
\begin{aligned}
T(n) &= aT(\frac{n}{b}) + n^c \\
&= a[aT(\frac{n/b}{b}) + (n/b)^c] + n^c \\
&= a^2T(\frac{n}{b^2}) + a(\frac{n}{b})^c + n^c \\
&= a^2[aT(\frac{n/b^2}{b}) + (n/b^2)^c] + a(\frac{n}{b})^c + n^c \\
&= a^3T(\frac{n}{b^3}) + a^2(\frac{n}{b^2})^c + a(\frac{n}{b})^c + n^c \\
&= \ldots \\
&= a^kT(\frac{n}{b^k}) + a^{k-1}(\frac{n}{b^{k-1}})^c + a^{k-2}(\frac{n}{b^{k-2}})^c + \ldots + a(\frac{n}{b})^c + n^c \\
&= T(1)\, a^{\lg_b n} + n^c \left[ \sum_{j=0}^{k-1} \left(\frac{a}{b^c}\right)^j \right] \\
&=^* T(1)\, n^{\lg_b a} + \begin{cases} n^c \lg_b n & \text{if } \; a = b^c \\ n^c \dfrac{1 - \left(\frac{a}{b^c}\right)^{\lg_b n}}{1 - \frac{a}{b^c}} & \text{if } \; a \neq b^c \end{cases}
\end{aligned}
$$

* Recall that $\sum_{j=0}^{k-1} \alpha^j = \begin{cases} k & \text{if } \; \alpha = 1 \\ \dfrac{1-\alpha^k}{1-\alpha} & \text{if } \; \alpha \neq 1 \end{cases}$

$$T(n) = \begin{cases} aT(\frac{n}{b}) + n^c & \text{if } n \geq b \\ \text{bounded} & \text{if } n < b \end{cases}$$

- if $a = b^c$ then $\lg_b a = c$ and

$$T(n) = T(1)\, n^{\lg_b a} + n^c \lg_b n \quad \in \Theta(n^c \lg n)$$

- if $a < b^c$ then $\lg_b a < c$ and

$$T(n) = T(1)\, n^{\lg_b a} + n^c \frac{1 - (a/b^c)^{\lg_b n}}{1 - a/b^c} \quad \in \Theta(n^c)$$

- if $a > b^c$ then $\lg_b a > c$ and

$$\begin{aligned} T(n) &= T(1)\, n^{\lg_b a} + n^c \frac{(a/b^c)^{\lg_b n} - 1}{(a/b^c) - 1} \\ &\in \Theta(\ n^{\lg_b a} + n^c \left(\frac{a}{b^c}\right)^{\lg_b n}\ ) \\ &\in \Theta(\ n^{\lg_b a} + n^c \frac{a^{\lg_b n}}{(b^c)^{\lg_b n}}\ ) \\ &\in \Theta(\ n^{\lg_b a} + n^c \frac{n^{\lg_b a}}{n^c}\ ) \\ &\in \Theta(\ n^{\lg_b a}\ ) \end{aligned}$$

$$T(n) = \begin{cases} aT(\frac{n}{b}) + n^c & \text{if } n \geq b \\ \text{bounded} & \text{if } n < b \end{cases}$$

- if $a = b^c$ then $\lg_b a = c$ and $T(n) \in \Theta(n^c \lg n)$
- if $a < b^c$ then $\lg_b a < c$ and $T(n) \in \Theta(n^c)$
- if $a > b^c$ then $\lg_b a > c$ and $T(n) \in \Theta( n^{\lg_b a} )$

- same result if $n^c$ replaced by $f(n) \in \Theta(n^c)$

- algorithm design consequences:
  - $\lg_b a = c$? balance: $T(n) \uparrow$ if $\lg_b a$ or $c \uparrow$
    redesign: no
  - $\lg_b a < c$? rec'n slack: $T(n) \downarrow$ if $c \downarrow \lg_b a \uparrow$
    redesign: less body, more/bigger calls
  - $\lg_b a > c$? body slack: $T(n) \downarrow$ if $\lg_b a \downarrow c \uparrow$
    redesign: fewer/smaller calls, more body

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some examples ...

$$T(n) = \begin{cases} 2T(n/2) + n & \text{if } n \geq 2 \\ T(1) & \text{if } n = 1 \end{cases}$$

$a = 2$, $b = 2$, $c = 1$, $a = b^c$ so $\qquad T(n) \in \Theta(n \lg n)$

$$T(n) = \begin{cases} 4T(n/2) + n^3 & \text{if } n \geq 2 \\ T(1) & \text{if } n = 1 \end{cases}$$

$a = 4$, $b = 2$, $c = 3$, $a < b^c$ so $\qquad T(n) \in \Theta(n^3)$

$$T(n) = \begin{cases} 7T(n/2) + n^2 & \text{if } n \geq 2 \\ T(1) & \text{if } n = 1 \end{cases}$$

$a = 7$, $b = 2$, $c = 2$, $a > b^c$ so $\qquad T(n) \in \Theta(n^{\lg_2 7})$

$$T(n) = \begin{cases} 7T(n/2) + n^2/\lg n & \text{if } n \geq 2 \\ T(1) & \text{if } n = 1 \end{cases}$$

$a = 7$, $b = 2$, $c =$??? ...so formula does not apply

exercise: in the last example, show that $T(n) \in O(n^{\lg_2 7})$