Lecture 7: Monday January 20, 2003

today

- Chapter 4: recurrences

announcements

- quizzes start today

Recurrence relations (a.k.a. recurrences) [CLRS Ch. 4]

- *recurrence relation*:

  a relation defined recursively, namely in terms of itself.
- must have base case and general case
- examples:
  - $f(n) = \begin{cases} 1 & \text{if } n = 1 \\ n + f(n-1) & \text{if } n \geq 2 \end{cases}$
  - $g(n) = \begin{cases} 1 & \text{if } n = 1 \\ n * g(n-1) & \text{if } n \geq 2 \end{cases}$
  - $t(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2 * t(\lfloor n/2 \rfloor) + n & \text{if } n \geq 2 \end{cases}$
- arise in analysis of D&C algorithms
- how are r.r. derived?
- how are r.r. solved? — by iterated substitution
  - particular cases: solve small examples exactly
  - general case: guess the answer, prove by induction
- [CLRS]

  §4.1 substitution method — a.k.a. iterated substitution

  §4.2 recursion tree method — ... with rec'n trees

  §4.3 master method — ... with master formula

iterated substitution: a really easy example

- $f(n) = \begin{cases} 1 & \text{if } \ n = 1 \\ n + f(n-1) & \text{if } \ n \geq 2 \end{cases}$

- particular cases:

n	1	2	3	4	5	6	7
f(n)	1	2+1	3+3	4+6	5+10	6+15	7+21
	1	3	6	10	15	21	28

- general case:

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f(n) = n + f(n-1)
     = n + n-1 + f(n-2)
     = n + n-1 + n-2 + f(n-3)
     = ...
     = n + n-1 + n-2 + ... + 3 + 2 + f(1)
     = n + n-1 + n-2 + ... + 3 + 2 + 1
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- "…" is not precise; replace above with summation formula

- guess answer: $f(n) =? \sum_{j=1}^{n} j$

- proof by induction

a really easy example (cont'd)

claim $f(n) = \sum_{j=1}^{n} j \qquad \forall n \geq 1$

proof by induction

base case

$$\begin{aligned} f(1) &= 1 && \text{by def'n} \\ &= \sum_{j=1}^{1} j && \text{so claim holds in this case} \end{aligned}$$

inductive case

assume claim holds for $n = k \geq 1$

want to show claim holds for $n = k + 1$

$$\begin{aligned} f(k+1) &= (k+1) + f(k+1-1) && \text{by def'n} \\ &= (k+1) + \sum_{j=1}^{k} j && \text{by ind. assumption} \\ &= \sum_{j=1}^{k+1} j && \text{by including k+1} \end{aligned}$$

so claim holds in this case, so claim holds for all $n \geq 1$ ☺

a really easy example (conclusion)

- have used iterated subst'n method to show $f(n) = \sum_{j=1}^{n} j$
- closed form solution is more useful
- using above, prove by induction that $f(n) = \dfrac{n(n+1)}{2}$

claim $f(n) = \dfrac{n(n+1)}{2} \qquad \forall n \geq 1$

proof by induction

base case

$$\begin{aligned} f(1) &= 1 && \text{by def'n} \\ &= \frac{1 \cdot 2}{2} && \text{so claim holds in this case} \end{aligned}$$

inductive case

assume claim holds for $n = k \geq 1$

want to show claim holds for $n = k + 1$

$$\begin{aligned} f(k+1) &= k + 1 + f(k) && \text{def'n} \\ &= k + 1 + \frac{k(k+1)}{2} && \text{ind. assumption} \\ &= (k+1)(1 + \frac{k}{2}) && \text{arithmetic} \\ &= \frac{(k+1)(k+2)}{2} && \text{arithmetic} \end{aligned}$$

so claim holds in this case, so claim holds for all $n \geq 1$ ☺

iterated substitution: another easy example

- $t(n) = \begin{cases} 1 & \text{if } \; n = 1 \\ 2 * t(\lfloor n/2 \rfloor) + n & \text{if } \; n \geq 2 \end{cases}$
- particular cases:

k	0	1	2	3	4	5
t(2^k)	1	2*1+2	2*4+4	2*12+8	2*32+16	2*80+32
	1	4	12	32	80	192

- general case:

$$\begin{aligned} t(2^k) &= 2 * t((2^k)/2) \; + \; 2^k \\ &= 2 * t(2^{k-1}) \; + \; 2^k \\ &= 2 * [2 * t((2^{k-1})/2) \; + \; 2^{k-1}] \; + \; 2^k \\ &= 2 * 2 * t(2^{k-2}) \; + \; 2^k \; + \; 2^k \\ &= 2 * 2 * [2 * (t((2^{k-2})/2) \; + \; 2^{k-2}] \; + \; 2^k \; + \; 2^k \\ &= 2 * 2 * 2 * (t(2^{k-3}) \; + \; 2^k \; + \; 2^k \; + \; 2^k \\ &= \quad \ldots \\ &= 2^k * t(2^{k-k}) \; + \; 2^k \; + \; 2^k \; + \; \ldots \; + \; 2^k \\ &= 2^k * 1 \; + \; k * 2^k \\ &= (k+1) * 2^k \end{aligned}$$

- guess: $t(2^k) =?(k+1)2^k$
- proof: by induction

claim $t(2^k) = (k+1)2^k$ $\qquad \forall k \geq 0$ $\qquad$ **proof** by induction

base case

$$\begin{aligned} t(2^0) &= 1 && \text{by def'n} \\ &= (0+1)2^0 && \text{so claim holds in this case} \end{aligned}$$

inductive case

assume claim holds for $k = p \geq 0$

want to show claim holds for $k = p+1$

$$\begin{aligned} t(2^{p+1}) &= 2\,t(2^{p+1}/2) + 2^{p+1} && \text{def'n} \\ &= 2\,t(2^p) + 2^{p+1} && \text{arithmetic} \\ &= 2\,(\,(p+1)2^p\,) + 2^{p+1} && \text{ind. assumption} \\ &= (p+1)\,2^{p+1} + 2^{p+1} && \text{arithmetic} \\ &= (p+1+1)\,2^{p+1} && \text{arithmetic} \\ &= (p+2)\,2^{p+1} && \text{arithmetic} \end{aligned}$$

so claim holds in this case, so claim holds for all $k \geq 0$ ☺