Lecture 6: Fri January 17, 2003

today

- sample answers to typical problems

announcements

- quizzes start Monday

4 typical problems

1. Prove the following pseudocode postcondition.

```
* postcondition:  A[1] is maximum among A[1..n]
for j <- n downto 2 do
  if  A[j] > A[j-1]  then
    interchange A[j] <-> A[j-1]
```

2. Prove that $\sum_{j=1}^{n} \lg j$ is in $\Theta(n \lg n)$.

3. Let $f(n)$ and $g(n)$ be non-negative functions,
let $s(n) = f(n) + g(n)$, and let $m(n) = \max\{f(n), g(n)\}$.
Prove or disprove: $s(n)$ is in $\Theta(m(n))$.

4. Using Θ notation, give the simplest description of the running time of the following, assuming a uniform RAM model of computation.

```
for j <- 1 to n-2 do
  for k <- j+1 to n-1 do
     for m <- k+1 to n do
       x <- x + a
```

an answer to 1.

- the key is to find a useful loop invariant
- LI: at start of 1, A[j] is max among A[j..n]
- now prove LI
- initialization: $j = n$, A[n] is max among A[n..n], so LI holds
- maintenance: assume LI holds for $j = t$, where $2 \leq t \leq n$
  - at start of 2, A[t] is max among A[t..n]
  - case 1: suppose A[t] $\leq$ A[t-1]
    - at end of 2, A[t-1] $\geq$ max among A[t..n], so A[t-1] max among A[t-1..n]
    - 3 does not execute, so next time 1 is reached

      A[t-1] is max among A[t-1..n] ☺
  - case 2: suppose A[t] $>$ A[t-1]
    - at end of 2, A[t] max among A[t..n] and $>$ A[t-1], so A[t] max among A[t-1..n]
    - 3 executes, so next time 1 is reached

      A[t-1] max among A[t-1...n] ☺
- init'n, maintenance, and induction imply that

  LI holds for all j with $1 \leq j \leq n$. ☺
- termination
  - execution reaches 1 for last time with j=1
  - LI for j=1 implies A[1] is max among A[1..n]
  - so postcondition holds ☺

an answer to 2. Let $s(n) = \sum_{j=1}^{n} \lg j$.

- for $n \geq 1$, $s(n) \;\leq\; \sum_{j=1}^{n} \lg n \;=\; n \lg n$, so $s(n) \in O(n \lg n)$. ☺

let $t = \lceil n/2 \rceil$

$$
\text{for } n \geq 2,\; s(n) \;\geq\; \sum_{j=t}^{n} \lg j
\;\geq\; \sum_{j=t}^{n} \lg t
\;\geq\; \frac{n-1}{2} \lg t
\;\geq\; \frac{n-1}{2} \lg(n/2)
$$

for $n > 4$, $\lg \frac{n}{2} > \frac{\lg n}{2}$

$$
\begin{aligned}
\text{so } s(n) \;&\geq\; \frac{n-1}{2} \left(\frac{\lg n}{2}\right) \\
&=\; \frac{n}{4} \lg n \;-\; \frac{\lg n}{4} \\
&=\; \frac{n}{8} \lg n \;+\; \frac{n}{8} \lg n \;-\; \frac{2 \lg n}{8} \\
&=\; \frac{n}{8} \lg n \;+\; \frac{n \lg n - 2 \lg n}{8} \\
&>\; \frac{n}{8} \lg n
\end{aligned}
$$

- so $s(n)$ is in $\Omega(n \lg n)$. ☺
- $s(n)$ in $O(n \lg n)$ and $\Omega(n \lg n)$, so $s(n)$ in $\Theta(n \lg n)$ ☺

an answer to 3.

- for each n,

  $f(n) \leq m(n)$, $g(n) \leq m(n)$, so $s(n) \leq 2m(n)$, so

  $s(n) \in O(m(n))$ ☺

- for each n,

  $m(n) = f(n)$ or $m(n) = g(n)$, $f(n) \geq 0$, $g(n) \geq 0$, so

  $s(n) \geq m(n)$, so $s(n) \in \Omega(m(n))$ ☺

- $s(n) \in O(m(n))$ and $s(n) \in \Omega(m(n))$ so $s(n) \in \Theta(m(n))$ ☺

an answer to 4.

- run time is in $\Theta(s(n))$ where $s(n) = \sum_{j=1}^{n-2} \sum_{k=j+1}^{n-1} \sum_{m=k+1}^{n} 1$
- $s(n) \le \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{m=1}^{n} 1 = \sum_{j=1}^{n} \sum_{k=1}^{n} n \;=\; \sum_{j=1}^{n} n^2 \;=\; n^3$, so $s(n) \in O(n^3)$

$$\begin{aligned}
s(n) \;&=\; \sum_{j=1}^{n-2} \sum_{k=j+1}^{n-1} (n-k) \\
&=\; \sum_{j=1}^{n-2} \left(n(n-1-j) - \sum_{k=j+1}^{n-1} k \right) \\
&=\; \sum_{j=1}^{n-2} \left(n(n-1-j) - (n-1-j)(n+j)/2 \right) \\
&=\; \frac{1}{2} \sum_{j=1}^{n-2} (n-1-j)(n-j) \\
&\ge\; \frac{1}{2} \sum_{j=1}^{t} (n-1-t)(n-t) \quad \text{for } t = \lfloor \frac{n}{2} \rfloor \\
&\ge\; \frac{1}{2} \, \frac{n-1}{2} \, \frac{n-2}{2} \, \frac{n}{2} \\
&\ge\; \frac{(n/2)^2 n}{16} \;=\; \frac{n^3}{64} \quad \text{for } n \ge 4
\end{aligned}$$

- so $s(n) \in \Omega(n^3)$, and $O(n^3)$, so $s(n) \in \Theta(n^3)$
- so run time in $\Theta(n^3)$

- note: can show $s(n) = \binom{n}{3} = n(n-1)(n-2)/6$
- note: can show $s(n)$ in $\Omega(n^3)$ using integration:
  $s(n) \ge \int_{j=1}^{n-1} \int_{k=j+1}^{n} \int_{m=k+1}^{n+1} 1 \, dm \, dk \, dj$