Lecture 5: Wednesday Jan 15, 2003

today

- growth of functions [CLRS Ch. 3]

announcements

- seminars start this week
- quiz 1 starts next week

our story so far . . .

- analysis of algorithms $\Rightarrow$ analysis of functions
- to simplify alg'm analysis, want function notation which indicates 'rate of growth' (a.k.a. 'order' of complexity)

O(f(n)) "big O of f(n)"

roughly: the set of functions which, as n gets large, grow no faster than a constant times $f(n)$

precisely: the set of functions $\{h(n) : N \mapsto R\}$ s.t. for each $h(n)$, there are $c_0 \in R^+$ and $n_0 \in N$ s.t. $h(n) \leq cf(n)$ for all $n \geq n_0$

$h(n) = 3n^2 + 10n + 1000 \lg n \in O(n^2)$**?**

$h(n) = 3n^2 + 10n + 1000 \lg n \in O(n^3)$**?**

$h(n) = \begin{cases} 5^n & n \leq 10^{100} \\ n^2 & n > 10^{100} \end{cases} \in O(n^2)$**?**

$O \quad \Omega \quad \Theta \quad o \quad \omega$

$h(n), f(n) : N \mapsto R$ $\qquad c, c_1, c_2 \in R^+ \ (> 0)$ $\qquad n_0 \in N \ (\geq 0)$

- $O(f(n))$ is the set of functions $h(n)$ that
  - roughly, grow **no faster than** $f(n)$, namely
  - $\exists\, c, n_0$ s.t. $h(n) \leq cf(n) \ \forall\, n \geq n_0$
- $\Omega(f(n))$ is the set of functions $h(n)$ that
  - roughly, grow **at least as fast as** $f(n)$, namely
  - $\exists\, c, n_0$ s.t. $h(n) \geq cf(n) \ \forall\, n \geq n_0$
- $\Theta(f(n))$ is the set of functions $h(n)$ that
  - roughly, grow **at the same rate as** $f(n)$, namely
  - $\exists\, c_1, c_2, n_0$ s.t. $c_1 f(n) \leq h(n) \leq c_2 f(n) \ \forall\, n \geq n_0$
  - are in $O(f(n))$ and $\Omega(f(n))$
- $o(f(n))$ is the set of functions $h(n)$ that
  - roughly, grow **more slowly than** $f(n)$, namely
  - $\lim_{n\to\infty} \frac{h(n)}{f(n)} = 0$
- $\omega(f(n))$ is the set of functions $h(n)$ that
  - roughly, grow **more quickly than** $f(n)$, namely
  - $\lim_{n\to\infty} \frac{h(n)}{f(n)} = \infty$
  - $h(n) \in \omega(f(n))$ if and only if $f(n) \in o(h(n))$

notice: this definition of o, ω simpler than in text

WARNING: the text overloads '='

- text: $f(n) = O(g(n))$
- incorrect, because $O(g(n))$ is a *set* of functions ☹
- correct: $f(n) \in O(g(n))$
- these notes: $f(n) \in O(g(n))$
- your work: $f(n) \in O(g(n))$

$f_1(n) = 19n \qquad f_2(n) = 77n^2 \qquad f_3(n) = 6n^3 + n^2 \ln n \qquad f_4(n) = 11n^4$

- in $O(n^3)$? $\quad f_1,\ f_2,\ f_3$

$$\begin{aligned} 19n &\le 19n^3 \ \ \forall n \ge 1 \\ 77n^2 &\le 77n^3 \ \ \forall n \ge 1 \\ 6n^3 + n^2 \ln n &\le 6n^3 + n^3 = 7n^3 \ \ \forall n \ge 1 \\ 11n^4 &\le cn^3 \Rightarrow n \le c/11 \end{aligned}$$

- in $\Omega(n^3)$? $\quad f_3,\ f_4$

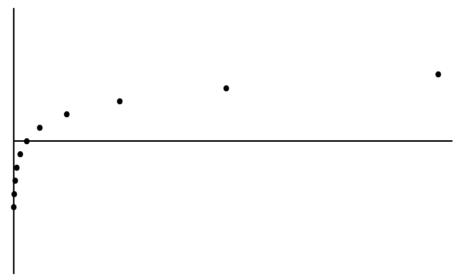
$$\begin{aligned} 19n &\ge cn^3 \Rightarrow 19/n^2 \ge c \\ 77n^2 &\ge cn^3 \Rightarrow 77/n \ge c \\ 6n^3 + n^2 \ln n &\ge n^3 \quad \forall n \ge 1 \\ 11n^4 &\ge n^3 \quad \forall n \ge 1 \end{aligned}$$

- in $\Theta(n^3)$? $\quad f_3$
- in $o(n^3)$? $\quad f_1,\ f_2$

$$\begin{aligned} \lim_{n\to\infty} 19n/n^3 &= \lim_{n\to\infty} 19/n^2 = 0 \\ \lim_{n\to\infty} 77n^2/n^3 &= \lim_{n\to\infty} 77/n = 0 \\ \lim_{n\to\infty}(6n^3 + n^2 \ln n)/n^3 &= \lim_{n\to\infty}(6n^3/n^3 + n^2 \ln n)/n^3) = 6 + 0 = 6 \\ \lim_{n\to\infty} 11n^4/n^3 &= \lim_{n\to\infty} 11n = \infty \end{aligned}$$

- in $\omega(n^3)$? $\quad f_4$

interlude: logarithm review text p15



- def'n $(b, n > 0)$: $b^{\log_b n} = n$
- $\log_b n$ increasing, 1-1
- $\log_b 1 = 0$ — $b^0 = 1$
- $\log_b x^p = p \log_b x$ — $(b^v)^w = b^{vw}$
- $\log_b(xy) = \log_b x + \log_b y$ — $b^v b^w = b^{v+w}$
- $x^{\log_b y} = y^{\log_b x}$ — hint: $z = b^{\log_b z}$
- $\log_b x = (\log_b c) \log_c x$ — hint: exponentiate base b

notes

- 18th century multiplication machine ☺
- $\ln n$: $\log_e n$ ('natural')
- $\lg n$: $\log_2 n$ (binary)
- $\Theta(\log_b n) = \Theta(\log_{whatever} n) = \Theta(\log n)$
- $\frac{d}{dx} \ln x = 1/x$
- $(\log n)^k \in o(n^\varepsilon)$ — $\forall\ k, \varepsilon > 0$

handy 'big O' tips ...

- $h \in O(f)$ if and only if $f \in \Omega(h)$ [different c]
- limit rule: if $\lim_{n\to\infty} \frac{h(n)}{f(n)} = \ldots$
  - $\ldots \quad \infty$ then $h \in \Omega(f)$
  - $\ldots O < k < \infty$ then $h \in \Theta(f)$
  - $\ldots \quad O$ then $h \in o(f)$
- L'Hôpital's rule: if
  - $\lim_{n\to\infty} h(n) = \infty$
  - $\lim_{n\to\infty} f(n) = \infty$
  - h' and f' exist
  - then $\lim_{n\to\infty} \frac{h(n)}{f(n)} = \lim_{n\to\infty} \frac{h'(n)}{f'(n)}$
  - e.g. $\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{1/n}{1} = 0$
- can't always use 'H rule
  - let $h(n) = \begin{cases} 1 & \text{if } n \text{ even} \\ n^2 & \text{if } n \text{ odd} \end{cases}$
  - $\lim_{n\to\infty} \frac{h(n)}{n^2}$ does not exist
  - $h(n) \in O(n^2), \Omega(1)$
- $O(\ )\ \Theta(\ )\ \Omega(\ )\ o(\ )\ \omega(\ )\ldots$ useful asymptotically only

analyze this...!

```
s <- 0
for j <- 1 to n do
    s <- s + 1 endfor
```

- model of computation? RAM
- what to count? copy, arith', branch tests
- total number of instructions? depends: how are numbers stored?
  - 1 number/cell:

$$t(n) \ = \ c + c + \ldots + c \ = \ \sum_{j=1}^{n} c \ = \ cn \ \in \Theta(n)$$

  - k bits/cell:

$$t(n) \ = \ \sum_{j=1}^{n} c\lceil(\lg(j+1))/k\rceil \ \in \Theta(n \log n)$$

analysis of $f(n) = \sum_{j=1}^{n} \lceil (\lg(j+1)) \rceil$

- upper bound?

$$\begin{aligned} f(n) &\le \sum_{j=1}^{n} (1 + \lg(n+1)) \\ &\le \sum_{j=1}^{n} (1 + 1 + \lg n) \\ &\le 2n + n \lg n \\ &\le (\lg n) n + n \lg n \qquad \forall n \ge 4 \\ &\le 2n \lg n \qquad \forall n \ge 4 \\ f(n) &\in O(n \log n) \end{aligned}$$

- lower bound? $f(n) \ \ge \ \sum_{j=1}^{n} \lg j$ so for even n

$$\begin{aligned} f(n) &\ge \sum_{j=n/2}^{n} \lg j \\ &\ge (n/2) \lg(n/2) \\ &\ge (n/2)(\lg n - 1) \\ &\ge (n/4)(2 \lg n - 2) \\ &\ge (n/4)(\lg n + \lg n - 2) \\ &\ge (n/4) \lg n \quad \forall n \ge 4 \end{aligned}$$

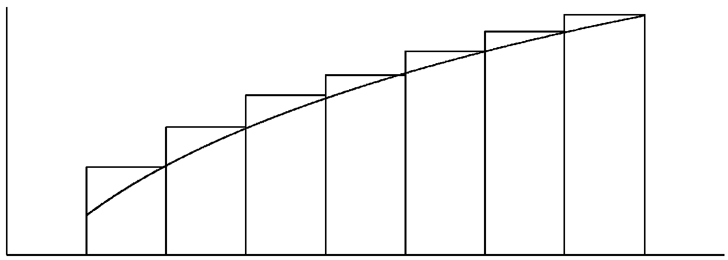
a similar argument holds for odd n, so

$$f(n) \in \Omega(n \log n)$$

- conclusion?

$$f(n) \in \Theta(n \log n)$$

- lower bound by integration



$$
\begin{aligned}
f(n) &\geq \sum_{j=2}^{n+1} \lg j \\
&\geq \int_1^{n+1} \lg x \, dx \\
&= \int_1^{n+1} (\lg e) \ln x \, dx \\
&= (\lg e)[x \ln x - x]_1^{n+1} \\
&= (\lg e)((n+1)\ln(n+1) - (n+1) - \ln 1 + 1) \\
&\in \Omega(n \log n)
\end{aligned}
$$

- another method: $\sum_{j=1}^{n} \lg j \quad = \lg(n!)$

  now use Stirling's approx'n: $\sqrt{2\Pi n}\left(\frac{n}{e}\right)^n \leq n! \leq \sqrt{2\Pi n}\left(\frac{n}{e}\right)^{13n/12}$