Lecture 35: Monday December 3, 2001

today

- NP completeness

announcements

- final exam
  - section A1/EA1 Tuesday Dec 11 9am-noon CSC B2
  - section A2/EA2 Tuesday Dec 11 9am-noon MEC 23
  - section A3/EA3 Tuesday Dec 18 2pm-5pm CSC B2

recall: basic concepts

- (abstract) problem
- decision problem
- the classes P, NP, co-NP
- the satifiability problem (SAT)
- completeness: the class NP-c
- Cook's theorem: SAT is in NP-c
- Karp's consequences: these problems are in NP-c
  - graph 3-colouring
  - graph Hamiltonian
  - ... (10 others)
- today: know thousands of problems $\in$ NP-c
- how to show a problem $\in$ NP-c

recall: how to show that a problem B is in NP-c

- use polynomial reduction (transformation)
  - show that B is in NP — usually easy
  - find an ‘old’ problem A (known to be in NP-c) — e.g. SAT
  - poly’ly reduce A to B
- consequence
  - any problem in NP can be reduced into A
  - A can be reduced into B . . . so
  - any problem in NP can be reduced into B . . . so
  - B is in NP-c
  - picture:

examples of polynomial reductions

- we will show
  - CNF-SAT reduces to 3-CNF-SAT — last class
  - 3-CNF-SAT reduces to 3-colouring — today
  - 3-colouring reduces to 4-colouring — today

recall: example reduction: CNF-SAT to 3-CNF-SAT

- transformation input: CNF-SAT formula f
- transformation output: 3-CNF-SAT formula f', such that

  f satisfiable iff f' satisfiable
- to obtain f' from f, for each clause

1 literal: $(z) \rightarrow$

$$(z \vee p \vee q) \wedge (z \vee p \vee \neg q) \wedge (z \vee \neg p \vee q) \wedge (z \vee \neg p \vee \neg q)$$

2 literals: $(y \vee z) \rightarrow$

$$(y \vee z \vee p) \wedge (y \vee z \vee \neg p)$$

3 literals: $(x \vee y \vee z) \rightarrow$

$$(x \vee y \vee z)$$

4 literals: $(v_1 \vee v_2 \vee v_3 \vee v_4) \rightarrow$

$$(v_1 \vee v_2 \vee p_1) \wedge (\neg p_1 \vee v_3 \vee v_4)$$

5 literals: $(v_1 \vee v_2 \vee v_3 \vee v_4 \vee v_5) \rightarrow$

$$(v_1 \vee v_2 \vee p_1) \wedge (\neg p_1 \vee v_3 \vee p_2) \wedge (\neg p_2 \vee v_4 \vee v_5)$$

...

consequence

- since
  - CNF-SAT is in NP-c (Cook proved this)
  - 3-CNF-SAT is in NP
  - CNF-SAT is polynomial-time-reducible to 3-CNF-SAT
- it follows that 3-CNF-SAT is in NP-c

example reduction: 3-colouring to 4-colouring

- transformation input: graph G
- transformation output: graph G', such that
  G 3-colourable   if and only if G' 4-colourable
- transformation: $G' \leftarrow G + h$, with h adj. to all $V(G)$
- observe
  - transformation takes polynomial time
  - any 3-colouring of G yields 4-colouring of G':
    use 4'th colour for h
  - any 4-colouring of G' yields 3-colouring of G:
    no vertex same colour of h, so only 3 colours on $G' - h$
  - so … polynomial reduction

consequence

- since
  - 3-colouring is in NP-c   we will show this later
  - 4-colouring is in NP
  - 3-colouring is polynomial-time-reducible to 4-colouring
- it follows that 4-colouring is in NP-c

example reduction: 3-CNF-SAT to 3-colouring

- transformation input: 3-CNF-SAT formula $f = C_1 \wedge C_2 \wedge \ldots \wedge C_m$ where variables $= x_1, \ldots, x_n$ and each $C_k = (C_{k,1} \vee C_{k,2} \vee C_{k,3})$
- transformation output: graph G_f, 3-colourable iff f sat'ble
- how to construct G_f from f:
  - vertices
    - t, f, g
    - x_j and $\overline{x}_j$ — for $j = 1$ to n
    - a_k, b_k, c_k, d_k, e_k — for $k = 1$ to m
  - edges
    - (tf) (tg) (fg)
    - (gx_j) and $(g\overline{x}_j)$ — for $j = 1$ to n
    - $(x_j\overline{x}_j)$ — for $j = 1$ to n
    - (a_kb_k) (a_kc_k) (b_kc_k) (c_kd_k) (d_ke_k) — for $k = 1$ to m
    - (td_k) (te_k) $(e_kC_{k,1})$ $(b_kC_{k,2})$ $(a_kC_{k,3})$ — for $k = 1$ to m

$f = (x_1\overline{x}_2x_3) \wedge (x_1x_2\overline{x}_4) \wedge (\overline{x}_1\overline{x}_3x_4) \wedge\ (x_2x_3\overline{x}_4) \wedge (\overline{x}_2\overline{x}_3x_4)$

missing edges?