Lecture 29: Monday November 19, 2001

today

- two MST implementations

announcements

- Assignment 8 due Monday November 26

recall: two greedy MST algorithms

- PDB (Prim/Dijkstra/Boruvka)

```
tree <- any vertex
do n-1 times
  add min cost edge e* which grows tree
```

- KB (Kruskal/Boruvka)

```
forest <- all vertices, no edges
do n-1 times
  add min cost edge f* which grows forest
```

- correctness uses graph theory
- different complexities, depend on imp'n

PDB MST simplest implementation

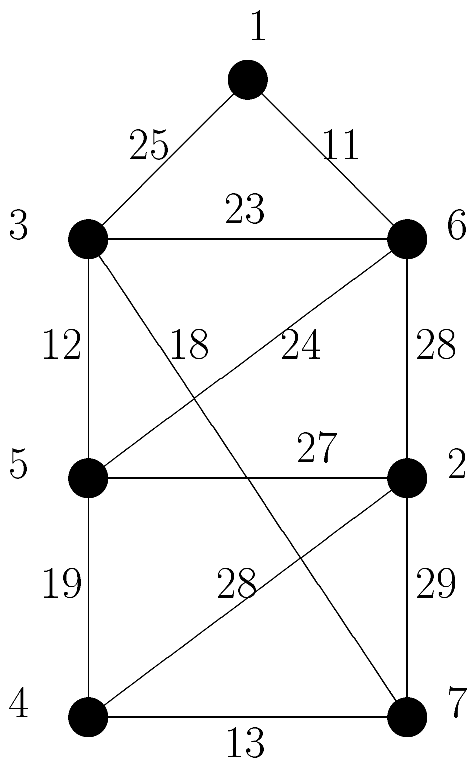
- find e* by examining all neighbours of all vertices
- run time in $\Theta(n^3)$ exercise

PDB MST better implementation

- idea 1: for each non-tree vertex, store min-cost tree neighbour
- idea 2: store values in priority queue
- to find e*:
  - ExtractMin(Q)
  - update 'best-neighbour' data

PDB MST implementation (generic) [CLRS p. 572]

```
MST(G,w,r)           { w: edge weights    r: starting vertex
for each u in V[G] do
  key[u]    <- infty     { min cost tree-neighbour from u
  parent[u] <- NIL
key[r] <- 0           { so r dequeued first
Q <- V[G]
while not empty(Q) do
  u <- ExtractMin(Q)
  for each v in Adj[u] do
   if v in Q and w(u,v) < key[v] then
     parent[v] <- u
     key[v]    <- w(u,v)   { requires Q adjustment
```




```
from 1
          1   2   3   4   5   6   7
Key       0   *   *   *   *   *   *    add   1   to tree
              *  25   *   *  11   *    add (1,6) to tree
             28  23   *  24       *    add (6,3) to tree
             28       *  12      18    add (3,5) to tree
             27      19          18    add (3,7) to tree
             27      13                add (7,4) to tree
             27                        add (5,2) to tree
```

PDB MST better imp'n: analysis

- assume
  - graph representation: adjacency list
  - priority queue imp'n: heap
- each ExtractMin $O(\lg n)$ total $\Theta(n \lg n)$
- for loop body
  - executed $\sum_u d(u)$ times total $\Theta(m)$
  - line 11 needs DecreaseKey total $\Theta(m \lg n)$
- total (for connected graph) $\Theta(n \lg n) + \Theta(m \lg n) = \Theta(m \lg n)$

MST algorithm correctness

- loop invariant for both algorithms:
  edges picked form subgraph of some MST [CLRS Thm. 23.1]
- proof, by induction, uses elementary graph theory see text

some graph theory used

- recall tree def'n: connected acyclic graph
- For a connected graph with n vertices,
  1. the graph is a tree (namely connected and acyclic) if and only if
  2. the graph is connected and has $n-1$ edges if and only if
  3. for each x, y there is a unique x, y-path.

proof sketch

- $1 \Longrightarrow 2$: in a tree, the ends of a maximal path have degree one; remove such a vertex, and argue by induction
- $1 \Longrightarrow 3$: suppose not; find a cycle
- $3 \Longrightarrow 1$: suppose not; find two paths
- $2 \Longrightarrow 1$: a connected graph with $m = n-1$ has no vertex with degree 0 (connected) so some vertex with degree 1 (sum the degrees); remove such a vertex, and argue by induction

detailed proof that 1 $\Rightarrow$ 2

- proofs on next page
- how many proofs can you do?

- *maximal* path: not a proper subpath
- *leaf*: degree 1 vertex
- lemma 1: in acyclic graph, end of maximal path has degree ≤ 1
- corollary: acyclic graph has vertex of degree 0 or 1
- lemma 2: tree with degree 0 vertex has only 1 vertex
- lemma 3: removing a leaf from a tree leaves a tree
- theorem: tree with n vertices has exactly $n - 1$ edges

- proof of lemma 1: if the end vertex of a path is adjacent to a vertex not in the path, the path is not maximal; if the end vertex p_0 of a path $(p_0, p_1, \ldots)$ is adjacent to a vertex p_j different from p_1, then $(p_0, p_1, \ldots, p_j, p_0)$ is a cycle.

- proof of lemma 2: in a connected graph there is a path between any two vertices, so if there is a vertex of degree 0 in a connected graph, it must be the only vertex in the graph.

- proof of lemma 3: Let T be a tree. Since T is acyclic, $T - v$ is acyclic.

  Let w be the neighbour of v in T. Let x, y be any two vertices of $T - v$. Since T is connected, there is a path $P = (x, \ldots, y)$ in T. Since every vertex in $P - \{x, y\}$ is adjacent to at least 2 path vertices, v is not in $P - \{x, y\}$, so v is not in P, so P is a path in $T - v$, so $T - v$ is connected.

- proof of theorem (by induction):

  base case: the graph with 1 vertex is a tree and has 0 edges

  inductive case: Fix $k \geq 1$ and assume that all trees with k vertices have $k - 1$ edges. Consider a tree T with $k + 1$ vertices. We want to show that T has k edges.

  Since $k \geq 1$, T has at least two vertices, so by lemmas 1 and 2, T has some vertex v with degree 1. By lemma 3, $T - v$ is a tree. $T - v$ has k vertices and (by inductive assumption) $k - 1$ edges. Since v is a leaf, T has exactly one more edge and one more vertex than $T - v$. Thus T has $1 + k - 1 = k$ edges.

KB MST algorithm

```
MST KB(G,w)
A <- empty   { edges of MST
for each v in V[G] do
  MakeSet(v)
sort edges of G into nondecreasing order by weight w
for each edge (u,v) (taken in order) do
  if Find(u) <> Find(v) then
    add (u,v) to A
    Union(u,v)
```

KB MST algorithm: analysis

- assume
  - G connected — so $m \geq n-1$
  - compressFind and rankUnion
- sort edges — total $\Theta(m \lg m) = \Theta(m \lg n)$

  since $m \in O(n^2$ implies $\lg m \in \Theta(\lg n)$
- n Makeset and $O(m)$ cF/rU — total $O((n+m)\alpha(n)) = O(m\alpha(n))$
- total $\Theta(m \lg n) + O(m\alpha(n)) = \Theta(m \lg m)$

  since $\alpha(n) \in O(\lg n)$