Lecture 8: Friday Sept 21, 2001

today

- an analysis Θ example: bounding summations
- Chapter 4: recurrences
  - more iterated substitution
  - recursion trees

announcements

- assignment 1 solutions (later today)
- assignment 2 due Monday in class

analyze this...!

```
s <- 0
for j <- 1 to n do
    s <- s + 1 endfor
```

- model of computation? RAM
- what to count? copy, arith', branch tests
- total number of instructions? depends: how are numbers stored?
  - 1 number/cell:

$$t(n) \; = \; c + c + \ldots + c \; = \; \sum_{j=1}^{n} c \; = \; cn \;\; \in \Theta(n)$$

  - k bits/cell:

$$t(n) \; = \; \sum_{j=1}^{n} c \lceil (\lg(j+1))/k \rceil \;\; \in \Theta(n \log n)$$

analysis of $f(n) = \sum_{j=1}^{n} \lceil (\lg(j+1)) \rceil$

- upper bound?

$$\begin{aligned} f(n) &\le \sum_{j=1}^{n} (1 + \lg(n+1)) \\ &\le \sum_{j=1}^{n} (1 + 1 + \lg n) \\ &\le 2n + n \lg n \\ &\le n \lg n + n \lg n \qquad \forall n \ge 2 \\ &\le 2n \lg n \qquad \forall n \ge 2 \\ f(n) &\in O(n \log n) \end{aligned}$$

- lower bound? $f(n) \ \ge \ \sum_{j=1}^{n} \lg j$ so for even n

$$\begin{aligned} f(n) &\ge \sum_{j=n/2}^{n} \lg j \\ &\ge (n/2) \lg(n/2) \\ &\ge (n/2)(\lg n - 1) \\ &\ge (n/4)(2 \lg n - 2) \\ &\ge (n/4)(\lg n + \lg n - 2) \\ &\ge (n/4) \lg n \quad \forall n \ge 4 \end{aligned}$$

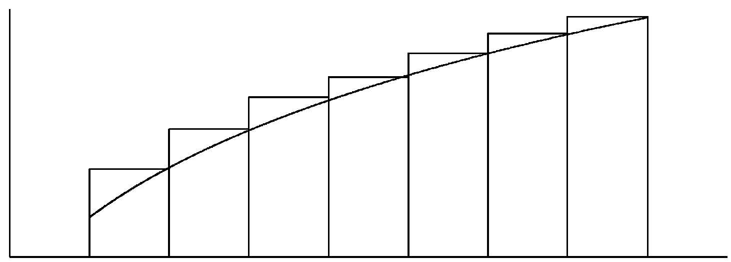
a similar argument holds for odd n, so

$$f(n) \in \Omega(n \log n)$$

- conclusion?

$$f(n) \in \Theta(n \log n)$$

- lower bound by integration



$$
\begin{aligned}
f(n) &\geq \sum_{j=2}^{n+1} \lg j \\
&\geq \int_1^{n+1} \lg x \, dx \\
&= \int_1^{n+1} (\lg e) \ln x \, dx \\
&= (\lg e)[x \ln x - x]_1^{n+1} \\
&= (\lg e)((n+1)\ln(n+1) - (n+1) - \ln 1 + 1) \\
&\in \Omega(n \log n)
\end{aligned}
$$

- another method: $\sum_{j=1}^{n} \lg j \ = \lg(n!)$

  now use Stirling's approx'n: $\sqrt{2\Pi n}\left(\frac{n}{e}\right)^n \leq n! \leq \sqrt{2\Pi n}\left(\frac{n}{e}\right)^{13n/12}$

our story so far

- D&C algorithm often recursive
- analysis of recursive algorithm $\Rightarrow$ recurrence relation
- example: running time of QZ(n)? (say uniform RAM)

```
proc QZ(n)
  if n > 1 then
    a <- n * n + 97
    b<-  a * QZ(n/2)
    return QZ(n/2) * QZ(n/2) + n
  else
    return n*n
```

- $A(n)$: during `QZ(n)`, number add'ns
- $M(n)$: during `QZ(n)`, number mult'ns
- $T(n)$: $A(n) + M(n)$
- claim: running time of `QZ(n)` $\in \Theta(T(n))$
- $A(n) =?$ $\quad M(n) =?$ $\quad T(n) =?$
- $T(n) = \begin{cases} 1 & \text{if} \quad n \leq 1 \\ 3\,T(\lfloor n/2 \rfloor) \, + \, 5 & \text{if} \quad n \geq 2 \end{cases}$
- solving $T(n)$?

solving $T(n)$ for $n = 2^k$

- repeated substitution
- see pattern
- guess answer
- prove answer by induction

$$
\begin{aligned}
T(n = 2^k) &= 3\,T(\frac{2^k}{2}) \;+5 && \text{def'n of } T(n) \\
&= 3\,T(2^{k-1}) \;+5 && \text{arith.} \\
&= 3\,(3\,T(\frac{2^{k-1}}{2}) + 5) \;+5 && \text{def'n of } T(n) \\
&= 3^2\,T(2^{k-2}) \;+3\cdot 5 \;+5 && \text{arith.} \\
&= 3^2\,(3\,T(\frac{2^{k-2}}{2}) \;+5) \;+3\cdot 5 \;+5 && \text{def'n of } T(n) \\
&= 3^3\,T(2^{k-3}) \;+3^2\cdot 5 \;+3\cdot 5 \;+5 && \text{arith.} \\
&= 3^3\,(3\,T(\frac{2^{k-3}}{2}) \;+5) \;+3^2\cdot 5 \;+3\cdot 5 \;+5 && \text{def'n of } T(n) \\
&= 3^4\,T(2^{k-4}) \;+\; (3^3 \;+\; 3^2 \;+\; 3^1 \;+\; 3^0)5 && \text{arith.} \\
&\ldots \\
&= 3^k\,(T(2^0)) \;+\; (3^{k-1} \;+\; \ldots \;+\; 3^1 \;+\; 3^0)5 && \text{guess pattern} \\
&= 3^k \;+\; \frac{3^k - 1}{3 - 1}\,5 && \text{geometric sum} \\
&= \frac{7}{2}3^k \;-\; \frac{5}{2} && \text{arith.}
\end{aligned}
$$