Lecture 7: Wed Sept 19, 2001

today

- Chapter 4: recurrences

announcements

- text
- assignment 2
- seminars

Recurrence relations (a.k.a. recurrences) [CLRS Ch. 4]

- *recurrence relation*:

  a relation defined recursively, namely in terms of itself.
- must have base case and general case
- examples:
  - $f(n) = \begin{cases} 1 & \text{if } n = 1 \\ n + f(n-1) & \text{if } n \geq 2 \end{cases}$
  - $g(n) = \begin{cases} 1 & \text{if } n = 1 \\ n * g(n-1) & \text{if } n \geq 2 \end{cases}$
  - $t(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2 * t(\lfloor n/2 \rfloor) + n & \text{if } n \geq 2 \end{cases}$
- arise in analysis of D&C algorithms
- how are r.r. derived?
- how are r.r. solved? by iterated substitution
  - particular cases: solve small examples exactly
  - general case: guess the answer, prove by induction
- [CLRS]

  §4.1 substitution method a.k.a. iterated substitution

  §4.2 recursion tree method ... with rec'n trees

  §4.3 master method ... with master formula

iterated substitution: a really easy example

- $f(n) = \begin{cases} 1 & \text{if } n = 1 \\ n + f(n-1) & \text{if } n \geq 2 \end{cases}$
- particular cases:

```
n        1     2     3     4     5     6     7
f(n)     1   2+1   3+3   4+6  5+10  6+15  7+21
         1     3     6    10    15    21    28
```

- general case:

```
f(n) = n + f(n-1)
     = n + n-1 + f(n-2)
     = n + n-1 + n-2 + f(n-3)
     = ...
     = n + n-1 + n-2 + ... + 3 + 2 + f(1)
     = n + n-1 + n-2 + ... + 3 + 2 + 1
```

- "..." is not precise; replace above with summation formula
- guess answer: $f(n) =? \sum_{j=1}^{n} j$
- proof by induction

a really easy example (cont'd)

claim $f(n) = \sum_{j=1}^{n} j \qquad \forall n \geq 1$

proof by induction

base case

$$\begin{aligned} f(1) &= 1 && \text{by def'n} \\ &= \sum_{j=1}^{1} j && \text{so claim holds in this case} \end{aligned}$$

inductive case

assume claim holds for $n = k \geq 1$

want to show claim holds for $n = k+1$

$$\begin{aligned} f(k+1) &= (k+1) + f(k+1-1) && \text{by def'n} \\ &= (k+1) + \sum_{j=1}^{k} j && \text{by ind. assumption} \\ &= \sum_{j=1}^{k+1} j && \text{by including k+1} \end{aligned}$$

so claim holds in this case, so claim holds for all $n \geq 1$ ☺

a really easy example (conclusion)

- have used iterated subst'n method to show $f(n) = \sum_{j=1}^{n} j$
- closed form solution is more useful
- using above, prove by induction that $f(n) = \dfrac{n(n+1)}{2}$

claim $f(n) = \dfrac{n(n+1)}{2} \qquad \forall n \geq 1$

proof by induction

base case

$$\begin{aligned} f(1) &= 1 && \text{by def'n} \\ &= \frac{1 \cdot 2}{2} && \text{so claim holds in this case} \end{aligned}$$

inductive case

assume claim holds for $n = k-1 \geq 1$

want to show claim holds for $n = k$

$$\begin{aligned} f(k) &= k + f(k-1) && \text{def'n} \\ &= k + \frac{(k-1)k}{2} && \text{ind. assumption} \\ &= k\left(1 + \frac{k-1}{2}\right) && \text{arithmetic} \\ &= \frac{k(k+1)}{2} && \text{arithmetic} \end{aligned}$$

so claim holds in this case, so claim holds for all $n \geq 1$ ☺

iterated substitution: another easy example

- $t(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2 * t(\lfloor n/2 \rfloor) + n & \text{if } n \geq 2 \end{cases}$
- particular cases:

k	0	1	2	3	4	5
t(2^k)	1	2*1+2	2*4+4	2*12+8	2*32+16	2*80+32
	1	4	12	32	80	192

- general case:

$$\begin{aligned}
t(2^k) &= 2 * t((2^k)/2) \ + \ 2^k \\
&= 2 * t(2^{k-1}) \ + \ 2^k \\
&= 2 * [2 * t((2^{k-1})/2) \ + \ 2^{k-1}] \ + \ 2^k \\
&= 2 * 2 * t(2^{k-2}) \ + \ 2^k \ + \ 2^k \\
&= 2 * 2 * [2 * (t((2^{k-2})/2) \ + \ 2^{k-2}] \ + \ 2^k \ + \ 2^k \\
&= 2 * 2 * 2 * (t(2^{k-3}) \ + \ 2^k \ + \ 2^k \ + \ 2^k \\
&= \quad \ldots \\
&= 2^k * t(2^{k-k}) \ + \ 2^k \ + \ 2^k \ + \ \ldots \ + \ 2^k \\
&= 2^k * 1 \ + \ k * 2^k \\
&= (k+1) * 2^k
\end{aligned}$$

- guess: $t(2^k) =?(k+1)2^k$
- proof: by induction

claim $t(2^k) = (k+1)2^k \qquad \forall k \geq 0$ **proof** by induction

base case

$$\begin{aligned} t(2^0) &= 1 && \text{by def'n} \\ &= (0+1)2^0 && \text{so claim holds in this case} \end{aligned}$$

inductive case

assume claim holds for $k = p - 1 \geq 0$

want to show claim holds for $k = p$

$$\begin{aligned} t(2^p) &= 2\,t(2^p/2) + 2^p && \text{def'n} \\ &= 2\,t(2^{p-1}) + 2^p && \text{arithmetic} \\ &= 2\,((p-1+1)2^{p-1}) + 2^p && \text{ind. assumption} \\ &= 2\,p2^{p-1} + 2^p && \text{arithmetic} \\ &= p\,2^p + 2^p && \text{arithmetic} \\ &= (p+1)\,2^p && \text{arithmetic} \end{aligned}$$

so claim holds in this case, so claim holds for all $k \geq 0$ ☺

our story so far

- D&C algorithm often recursive
- analysis of recursive algorithm $\Rightarrow$ recurrence relation
- example: running time of QZ(n)? (say uniform RAM)

```
proc QZ(n)
  if n > 1 then
    a <- n * n + 97
    b<-  a * QZ(n/2)
    return QZ(n/2) * QZ(n/2) + n
  else
    return n*n
```

- $A(n)$: during `QZ(n)`, number add'ns
- $M(n)$: during `QZ(n)`, number mult'ns
- $T(n)$: $A(n) + M(n)$
- claim: running time of `QZ(n)` $\in \Theta(T(n))$
- $A(n) =?$ $\quad M(n) =?$ $\quad T(n) =?$
- $T(n) = \begin{cases} 1 & \text{if} \quad n \leq 1 \\ 3\,T(\lfloor n/2 \rfloor) \,+\, 5 & \text{if} \quad n \geq 2 \end{cases}$
- solving $T(n)$?