Lecture 6: Mon Sept 17, 2001

today

- review of O Θ Ω o

announcements

- other sections start Ch. 3 today, finish Wednesday
- our section reviews Ch. 3 today, finish Wednesday
- next assignment next Monday
- acknowledgement on every assignment

recall: analysis issues

- issues
  - correctness
  - resources used (time/space)
  - simplicity
  - optimality
- estimating resources used
  - various input sizes $\Rightarrow t(n)\ s(n)$
  - worst/average/best cases (WC/AC/BC)
  - machine independence $\Rightarrow$ model
- model of computation
  - simple (architecture/instruction set)
  - reflective ('typical machine'; accurate estimates)
  - our choice: RAM (random access machine)
  - two versions: uniform cost, log cost
  - choose version depending on application
  - astronomical numbers: log cost
  - small numbers: uniform cost

when presenting analysis

for algorithm run times, remember to state what you have counted (e.g. RAM instructions? log RAM instructions? data moves? data comparisons? arithmetic operations? Pentium II clock cycles? etc.)

our story so far . . .

- analysis of algorithms $\Rightarrow$ analysis of functions
- 'order' of complexity (growth rate: dominant term, ignore const. co-eff.)

$h(n), f(n) : N \mapsto R^* \ (\geq 0) \qquad c, c_1, c_2 \in R^+ \ (> 0) \qquad n_0 \in N \ (\geq 0)$

- $O(f(n))$ is the set of functions $h(n)$ that
  - grow **no faster than** $f(n)$, namely
  - $\exists\, c, n_0$ s.t. $h(n) \leq cf(n) \ \forall\, n \geq n_0$
- $\Omega(f(n))$ is the set of functions $h(n)$ that
  - grow **at least as fast as** $f(n)$, namely
  - $\exists\, c, n_0$ s.t. $h(n) \geq cf(n) \ \forall\, n \geq n_0$
- $\Theta(f(n))$ is the set of functions $h(n)$ that
  - grow **at the same rate as** $f(n)$, namely
  - are in $O(f(n))$ and $\Omega(f(n))$
  - $\exists\, c_1, c_2, n_0$ s.t. $c_1 f(n) \leq h(n) \leq c_2 f(n) \ \forall\, n \geq n_0$
- $o(f(n))$ is the set of functions $h(n)$ that
  - grow **more slowly than** $f(n)$, namely
  - $\lim_{n\to\infty} \frac{h(n)}{f(n)} = 0$

WARNING: the text overloads '='

- text: $f(n) = O(g(n))$
- incorrect, because $O(g(n))$ is a *set* of functions
- correct: $f(n) \in O(g(n))$
- these notes: $f(n) \in O(g(n))$
- your work: $f(n) \in O(g(n))$

$f_1(n) = 19n \qquad f_2(n) = 77n^2 \qquad f_3(n) = 6n^3 + n^2 \ln n \qquad f_4(n) = 11n^4$

- in $O(n^3)$? $\qquad f_1,\ f_2,\ f_3$

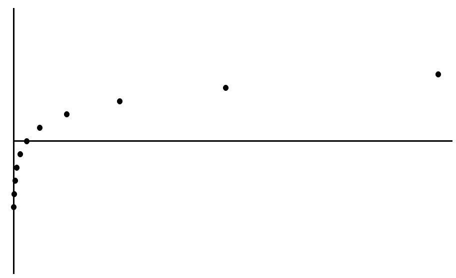
$$\begin{aligned} 19n &\le 19n^3 \quad \forall n \ge 0 \\ 77n^2 &\le 77n^3 \quad \forall n \ge 0 \\ 6n^3 + n^2 \ln n &\le 6n^3 + n^3 = 7n^3 \quad \forall n \ge 0 \\ 11n^4 &\le cn^3 \Rightarrow n \le c/11 \end{aligned}$$

- in $\Omega(n^3)$? $\qquad f_3,\ f_4$

$$\begin{aligned} 19n &\ge cn^3 \Rightarrow 19/n^2 \ge c \\ 77n^2 &\ge cn^3 \Rightarrow 77/n \ge c \\ 6n^3 + n^2 \ln n &\ge n^3 \quad \forall n \ge 1 \\ 11n^4 &\ge n^3 \quad \forall n \ge 0 \end{aligned}$$

- in $\Theta(n^3)$? $\qquad f_3$
- in $o(n^3)$? $\qquad f_1,\ f_2$

$$\begin{aligned} \lim_{n\to\infty} 19n/n^3 &= \lim_{n\to\infty} 19/n^2 = 0 \\ \lim_{n\to\infty} 77n^2/n^3 &= \lim_{n\to\infty} 77/n = 0 \\ \lim_{n\to\infty} (6n^3 + n^2 \ln n)/n^3 &= \lim_{n\to\infty} (6n^3/n^3 + n^2 \ln n)/n^3) = 6 + 0 = 6 \\ \lim_{n\to\infty} 11n^4/n^3 &= \lim_{n\to\infty} 11n = \infty \end{aligned}$$



- def'n $(b, n > 0)$: $b^{\log_b n} = n$
- $\log_b n$ increasing, 1-1
- $\log_b 1 = 0$ — $b^0 = 1$
- $\log_b x^p = p \log_b x$ — $(b^v)^w = b^{vw}$
- $\log_b(xy) = \log_b x + \log_b y$ — $b^v b^w = b^{v+w}$
- $x^{\log_b y} = y^{\log_b x}$ — hint: $z = b^{\log_b z}$
- $\log_b x = (\log_b c) \log_c x$ — hint: exponentiate base b

notes

- 18th century multiplication machine ☺
- $\ln n$: $\log_e n$ ('natural')
- $\lg n$: $\log_2 n$ (binary)
- $\Theta(\log_b n) = \Theta(\log_{whatever} n) = \Theta(\log n)$
- $\frac{d}{dx} \ln x = 1/x$
- $(\log n)^k \in o(n^\varepsilon)$ — $\forall\ k, \varepsilon > 0$

handy 'big O' tips ...

- $h \in O(f)$ if and only if $f \in \Omega(h)$ [different c]
- limit rule: if $\lim_{n\to\infty} \frac{h(n)}{f(n)} = \ldots$
  - ... ∞ then $h \in \Omega(f)$
  - ... $O < k < \infty$ then $h \in \Theta(f)$
  - ... O then $h \in o(f)$
- L'Hôpital's rule: if
  - $\lim_{n\to\infty} h(n) = \infty$
  - $\lim_{n\to\infty} f(n) = \infty$
  - h' and f' exist
  - then $\lim_{n\to\infty} \frac{h(n)}{f(n)} = \lim_{n\to\infty} \frac{h'(n)}{f'(n)}$
  - e.g. $\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{1/n}{1} = 0$
- can't always use 'H rule
  - let $h(n) = \begin{cases} 1 & \text{if } n \text{ even} \\ n^2 & \text{if } n \text{ odd} \end{cases}$
  - $\lim_{n\to\infty} \frac{h(n)}{n^2}$ does not exist
  - $h(n) \in O(n^2), \Omega(1)$
- $O(\ )\ \Theta(\ )\ \Omega(\ )\ o(\ )$ ... useful asymptotically only