Properties of Logarithms

The *logarithm* to the base b of n, where b is a real number > 1 and n is a natural number, is the power to which b must be raised to equal n. That is,

$$\begin{aligned} \log_b n &= k \quad \text{where } b^k = n \\ \text{and} \qquad n &= b^{\log_b n} \end{aligned}$$

We use a shorthand notation for logarithms to the base 2:

$$\lg n = \log_2 n$$

For real $a, b, c > 0$ and natural number n, where logarithm bases are greater than 1:

1. $\log_c(ab) = \log_c a + \log_c b$

   Proof.

$$\begin{aligned} \log_c(ab) &= \log_c(c^{\log_c a} \cdot c^{\log_c b}) \quad \text{because } a = c^{\log_c a} \text{ and } b = c^{\log_c b} \\ &= \log_c(c^{\log_c a + \log_c b}) \\ &= \log_c a + \log_c b \end{aligned}$$

2. $\log_c(a^n) = n \log_c a$

   Proof follows from 1.

3. $\log_c(1/a) = -\log_c a$

4. $\log_c(a/b) = \log_c a - \log_c b$

5. $\log_c n = \log_b n \cdot \log_c b$

   Proof.

$$\begin{aligned} \log_c n &= \log_c(b^{\log_b n}) \quad \text{because } n = b^{\log_b n} \\ &= \log_b n \cdot \log_c b \quad \text{by 2.} \end{aligned}$$

6. $a^{\log_b c} = c^{\log_b a}$

   Proof.

$$\begin{aligned} a^{\log_b c} &= (b^{\log_b a})^{\log_b c} \quad \text{because } a = b^{\log_b a} \\ &= b^{\log_b a \cdot \log_b c} \\ &= b^{\log_b c \cdot \log_b a} \\ &= (b^{\log_b c})^{\log_b a} \\ &= c^{\log_b a} \quad \text{because } c = b^{\log_b c} \end{aligned}$$