Runtime of recursive Fibonacci computation

Define $c(0) = 1$, $c(1) = 1$, and for $n \geq 2$, $c(n) = 1 + c(n-1) + c(n-2)$

Claim For all $n \geq 1$, $c(n) \geq t1.6^n$, where t is a constant

Proof by induction

Base case below

Inductive case Let $k \geq 2$ be an integer. Assume claim holds for all non-negative integers less than k.

Now

$$
\begin{aligned}
c(k) &= 1 + c(k-1) + c(k-2) && \text{by definition of c()} \\
&> c(k-1) + c(k-2) && \text{drop the 1} \\
&\geq t1.6^{k-1} + t1.6^{k-2} && \text{inductive assumption} \\
&= t(1.6+1)1.6^{k-2} && \text{arithmetic} \\
&= t(2.6)1.6^{k-2} && \text{arithmetic} \\
&> t(1.6^2)1.6^{k-2} && \text{since} 2.6 > 2.56 = 1.6^2 \\
&= t1.6^{k-2+2} = t1.6^k && \text{arithmetic}
\end{aligned}
$$

So assuming that the claim holds for non-negative integers $n < k$ implies that the claim also holds for $n = k$, so the claim holds for all non-negative integers (once we prove that the claim holds in the base case).

Base case What should t be so claim holds for $n \leq 1$? For $n = 0$, $c(n) = 1$, and $1.6^0 = 1$, so here claim holds as long as as $t \leq 1$. For $n = 1$, $c(n) = 1$, and $1.6^n = 1.6$, so here the claim holds as long as $t \leq 1/1.6 = 5/8$. So set t to minimum of these two values, so $5/8$, and now the claim holds for all $n \geq 0$.

Corollary $c(n) \in \Omega(1.6^n)$ Let $\alpha = (1+\sqrt{5})/2$ **Remark**: also $c(n) \in \Omega(\alpha^n)$

Exercise exists t, $c(n) \leq t1.7^n - 1$, so $c(n) \in O(1.7^n)$ **Remark**: also $O(\alpha^n)$

Corollary to remarks $c(n) \in \Theta(\alpha^n)$