CS 170 Fall 2006 — Solutions to Chapter 7

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7.1 The optimal solution is achieved in the upper right corner of the convex feasible region, i.e. at the point $(5, 2)$, and has value $5x + 3y = 31$.

7.2 We will use concatenation of the first letters of the two cities for the shnupells of duckwheat transported between those cities (i.e. MN for the quantity of shnupells between Mexico and New York e.t.c.). The linear programm will be the following:
$\min 4MN + MC + 2KN + 3KC$
$MN + KN = 10$
$MC + KC = 13$
$MN + MC = 8$
$KN + KC = 15$
$MN, MC, KN, KC \geq 0$

7.3 Let q_i denote the quantity (in qubic meters) of material i. The linear program will be the following :
$\max 1,000q_1 + 1,200q_2 + 12,000q_3$
$2q_1 + q_2 + 3q_3 \leq 100$
$q_1 + q_2 + q_3 \leq 60$
$q_1 \leq 40$
$q_2 \leq 30$
$q_3 \leq 20$
$q_1, q_2, q_3 \geq 0$

7.4 Let R the quantity of RegularDuff beer and S of DuffStrong beer that Moe is ordering per week.
$\max R + 1.5S$
$S \leq 2R$
$R + S \leq 3,000$
$R, S \geq 0$

It is easily seen geometrically that the maximum is obtained at the vertex $(R, S) = (0, 3, 000)$ and optimum $= 45000$.

7.5 Let F be the number of packages of Frisky Pup and H of Husky Hound.
(a) $\max 7F + 6H - F - 2 \cdot 1.5F - 2H - 2H - 1.4F - 0.6H = 1.6F + 1.4H$

$F + 2H \leq 240,000$
$1.5F + H \leq 180,000$
$F \leq 110,000$
$F, H \geq 0$

(b) ??

7.6

$$\begin{aligned} \max \quad & x - 2y \\ x - y &\leq 1 \\ x, y &\geq 0 \end{aligned}$$

7.7 a) This LP is never infeasible as the origin will satisfy $ax + by \leq 1$ for any choice of a and b.

b) It is sufficient that $a \leq 0$ or $b \leq 0$. If $a \leq 0$, then we can increase x (and the objective function) arbitrarily without violating any constraint. The same argument works for b and y. Conversely, suppose both a and b are positive. Let $m = \min\{a, b\}$ and notice $m > 0$. Then, $m(x + y) \leq ax + by \leq 1$, so that $x + y \leq 1/m$. Hence, the LP cannot be unbounded.

c) By a) and b), the LP has a finite optimal when a and b are positive. Suppose now $a > b$. Then, the optimal is clearly uniquely achieved at $x = \frac{1}{b}$. Similarly, if $b > a$. the unique optimum is $x = \frac{1}{a}$. However, if $a = b$, then any positive pair (x, y) such that $x + y = \frac{1}{a}$ achieves the optimum. Hence, the optimum exists and is unique if and only if a, b are positive and $a \neq b$.

7.8 Let $z = \max_{1 \leq i \leq 7} |ax_i + by_i - c|$. Then the linear program can be written as follows:
$\min z$
$z \geq a + 3b - c$
$z \geq c - a - 3b$
$z \geq 2a + 5b - c$
$z \geq c - 2a - 5b$
$z \geq 3a + 7b - c$
$z \geq c - 3a - 7b$
$z \geq 5a + 11b - c$
$z \geq c - 5a - 11b$
$z \geq 7a + 14b - c$
$z \geq c - 7a - 14b$
$z \geq 8a + 15b - c$
$z \geq c - 8a - 15b$
$z \geq 10a + 19b - c$
$z \geq c - 10a - 19b$
$z \geq 0$

7.9 Consider the following quadratic program :
$\max x_1 x_2$
$x_1 + x_2 \leq 1$
$x_1, x_2 \geq 0$

It is easy to see that the maximum is obtained at the point $x_1 = x_2 = 1/2$ whereas the vertices of the feasible region are $(0,0), (1,0)(0,1)$.

7.10 We run the simplex algorithm as in the paradigm of figure 7.6 in the book, each time modifying the flow and the residual network. We end up with a maximum flow of 13 units, which corresponds to the cut $\{S, C, F\}$ and $\{A, B, D, E, G, T\}$.

7.11 The dual LP is:

$$\begin{array}{rcl} \max\ 3u & + & 5w \\ 2u + w & \geq & 1 \\ u + 3w & \geq & 1 \\ u, w & \geq & 0 \end{array}$$

The optimal solution for the primal is $\frac{11}{5}$ given by $(x, y) = \left(\frac{4}{5}, \frac{7}{5}\right)$. The corresponding dual optimum is given by $(u, w) = \left(\frac{2}{5}, \frac{1}{5}\right)$.

7.12 Multiply the first inequality by y_1, the second by y_2 (with $y_1, y_2 \geq 0$) and add to obtain

$$(y_1)x_1 + (2y_2 - y_1)x_2 + (-y_2)x_3 \leq y_1 + y_2$$

To obtain the tightest constraint on the given objective function, we would like to obtain y_1 and y_2, subject to

$$\begin{array}{rr} \min & y_1 + y_2 \\ & y_1 \geq 1 \\ & 2y_2 - y_1 \geq 0 \\ & -y_2 \geq -2 \\ & y_1, y_2 \geq 0 \end{array}$$

This is the *dual* of the given linear program. Solving gives $y_1 = 1, y_2 = 1/2$. Plugging these values into the above inequality gives $x_1 - 1/2x_3 \leq 3/2$ which implies $x_1 - 2x_3 \leq 3/2$ since $x_3 \geq 0$. Finally, it remains to note that the given solution has value $3/2$ and is hence optimal.

7.13 a) The following matrix represents the payoffs of the game. Each entry gives R's revenue for that play.

		C	
		Head	Tail
R	Head	+1	-1
	Tail	-1	+1

b) We can obtain the value of the game by solving the finally LP for the optimal mixed strategy of R:

$$\begin{aligned} &\max\ z \\ z &\leq x_1 - x_2 \\ z &\leq x_2 - x_1 \\ x_1 + x_2 &= 1 \\ x_1, x_2 &\geq 0 \end{aligned}$$

This clearly has optimal value 0 given by $x_1 = x_2 = \frac{1}{2}$. Hence, the value of the game is 0.

7.14 Let Joey choose the two possible options with probabilities x_1 and x_2. Then the gains to Joey (or losses to Tony) corresponding to the three strategies of Tony are $\{2x_1 - x_2, -2x_2, -3x_1 + x_2\}$. Of course, Tony will choose the option that minimizes his loss and hence Joey will try to maximize the minimum of these three options. Thus, the optimal strategy for Joey can be obtained from the solution of the linear program

$$\begin{array}{rcl} \max z, & & \text{subject to} \\ z & \leq & 2x_1 - x_2 \\ z & \leq & -x_2 \\ z & \leq & -3x_1 + x_2 \\ x_1 + x_2 & = & 1 \\ x_1, x_2 & \geq & 0 \end{array}$$

Similarly, if Tony chooses his three strategies with probabilities y_1, y_2 and y_3, then Joey gains $\{2y_1 - 3y_3, -y_1 - 2y_2 + y_3\}$ by his two strategies and Tony tries to minimize the maximum gain. The corresponding linear program is

$$\begin{array}{rcl} \min z, & & \text{subject to} \\ z & \geq & 2y_1 - 3y_3 \\ z & \geq & -y_1 - 2y_2 + y_3 \\ y_1 + y_2 + y_3 & = & 1 \\ y_1, y_2, y_3 & \geq & 0 \end{array}$$

The solution to the first linear program is $x_1 = x_2 = 1/2$. For the second linear program $y_1 = 0, y_2 = 2/3, y_3 = 1/3$. The value of the game, which is the value of z in both the programs, is -1 i.e. Joey will loose the sale of