Define α as the golden ratio, i.e. $(1+\sqrt{5})/2$.

Define $f(n)$ as the nth Fibonacci number:

$f(0) = 0$, $f(1) = 1$, and, for all integers $t \geq 2$, $f(t) = f(t-1) + f(t-2)$.

Define $T(n)$ so that $T(0) = T(1) = 4$ and, for all $n \geq 2$, $T(n) = T(n-1) + T(n-2) + 7$.

1. **Prove, for all integers $n \geq 1$, that $\lg n < n$.**

   Let $P(n)$ be the above predicate, i.e. $\lg n < n$. Exponentiating both sides of the inequality with base 2, we see that $P(n)$ is equivalent to $Q(n) : n < 2^n$ (since $2^{\lg n} = n$). We can prove $Q(n)$ by induction.

   Base case: Let $n = 1$. Then $n = 1 < 2 = 2^1 = 2^n$, so $Q(1)$.

   Inductive case: Let t be any integer greater than 1. Assume $Q(n)$ for all integers n in the set $\{1, \ldots, t-1\}$. Under this assumption, we want to show $Q(t)$, i.e. that $t < 2^t$.

   But $t = (t-1) + 1$, and $2^t = 2 * (2^{t-1})$, so we have

   $t = (t-1) + 1 < 2^{t-1} + 1$ since $Q(t-1)$ by the induction hypothesis, and

   $2^{t-1} + 1 < 2^{t-1} + 2^{t-1} = 2^t$, since $1 < 2^{t-1}$ (because $t > 1$), so

   $t = (t-1) + 1 < 2^{t-1} + 1 < 2^t$, so $t < 2^t$, i.e. $Q(t)$.

   So, by the principle of mathematical induction, $Q(n)$ holds for all $n \geq 1$.

2. **Prove that $16n^2 + 99n + 16n \lg n$ is in $\Theta(n^2)$.**

   Define $q(n) = 16n^2 + 99n + 16n \lg n$.

   For $n \geq 1$, $99n > 0$ and $16n \lg n \geq 0$, so $q(n) > 16n^2$, so there exists a constant c (e.g. 16) such that $q(n) > cn^2$, so $q(n) \in \Omega(n^2)$.

   Also, for $n \geq 1$, $n \leq n^2$, so $99n \leq 99n^2$, and $\lg n < n \leq n^2$ (by the previous question), so $q(n) < 16n^2 + 99n^2 + 16n^2 = 131n^2$, so there exists a constant d (e.g. 131) such that $q(n) < dn^2$, so $q(n) \in O(n^2)$.

   Finally, $q(n) \in \Omega(n^2)$, and $q(n) \in O(n^2)$, so $q(n) \in \Theta(n^2)$.

3. **Prove, for all $n \geq 0$, $f(n) < \alpha^n$.**

   Let $P(n)$ be the above predicate, i.e. $f(n) < \alpha^n$. Prove the claim by induction on n.

   Base cases. Left as an exercise.

   Inductive case. Let k be any integer ≥ 2. Assume $P(n)$ for all integers n in the set $\{1, \ldots, k-1\}$. We want to show that this implies $P(k)$, i.e. that $f(k) < \alpha^k$.

   Now, $f(k) = f(k-1) + f(k-2)$ (by the definition of $f(k)$, since $k \geq 3$), and $P(k-1)$ and $P(k-2)$ by the inductive assumption, i.e. $f(k-1) < \alpha^{k-1}$ and $f(k-2) < \alpha^{k-2}$, so we have $f(k) < \alpha^{k-1} + \alpha^{k-2}$. So we are done if we can show $\alpha^{k-1} + \alpha^{k-2} \leq \alpha^k$, i.e. if

$$\alpha^k - \alpha^{k-1} - \alpha^{k-2} \geq 0 \,. \qquad (*)$$

But $\alpha^k - \alpha^{k-1} - \alpha^{k-2} = \alpha^{k-2}(\alpha^2 - \alpha - 1)$, and $\alpha^2 - \alpha - 1 = 0$ (start with the definition of α, and use arithmetic; alternatively, by the quadratic equation, $y^2 - y - 1$ is zero if $y = (1+\sqrt{5})/2$, i.e. if $y = \alpha$), so (*) holds, so we have shown $f(k) < \alpha^k$. So $P(k)$. So, by the principle of mathematical induction, $P(n)$ holds for all positive integers n.

4. The proof is similar to that of the previous question.

5. (i) **Prove, for all integers** $n \geq 0$**,** $T(n) = 11f(n+1) - 7$.

   Argue by induction. Define $P(n)$ as the above predicate. There are two bases to consider. I leave these to you as an exercise.

   Inductive case. Let x be a positive integer and assume $P(n)$ for all n in $\{1, \ldots, x-1\}$. We want to show $P(x)$, i.e. that $T(x) = 11f(x) - 7$.

   Now $T(x) = T(x-1) + T(x-2) + 7$ (why?), $T(x-1) = f(x) - 7$ (why?), and $T(x-2) = f(x-1) - 7$ (why?), and $f(x) + f(x-1) = f(x+1)$ (why?), so

$$\begin{array}{rcrcr} T(x) & = & T(x-1) & + & T(x-2) + 7 \\ & = & f(x) - 7 & + & f(x-1) - 7 + 7 \\ & = & f(x) & + & f(x-1) - 7 \\ & = & f(x+1) - 7 & & \end{array}$$

   so $P(x)$. So, by the principle of mathematical induction, $P(n)$ for all positive integers n.

   (ii) **Prove that** $T(n)$ **is in** $O(\alpha^n)$.

   Use (i) and question 3.

   (iii) **Prove that** $T(n)$ **is in** $\Theta(\alpha^n)$.

   Show $T(n)$ is in $\Omega(\alpha^n)$: prove $T(n) > f(n)$ by induction and use question 4. Now use (ii).