RN, Chapter 9

Predicate Calculus -- Inference

Logical Agents

- Reasoning [Ch 6]
- Propositional Logic [Ch 7]
- Predicate Calculus
  - Representation [Ch 8]
    - Syntax, Semantics, Expressiveness
    - Example: Circuits
  - Inference [Ch 9]
    - Resolution
- Implemented Systems [Ch 10]
- Planning [Ch 11]

Proof Process for Predicate Calculus

- Why not “Model Checking” approach?
- Still use

  Inference Rules

  to obtain new facts from exisiting info
- Still seeking...

If $\vdash$ is SOUND+COMPLETE,

$\Rightarrow \quad \vdash \equiv \models$

$\Rightarrow$ Computer can IGNORE SEMANTICS and just push symbols!

- Still exploiting

  SOUND inference rules

  MONOTONICITY

(Sound) Inference Rules

[MP]
$$\frac{\begin{array}{l} P \Rightarrow Q \\ P \end{array}}{Q}$$

[∀E]
$$\frac{\forall x.\, \varphi(x)}{\varphi(\mathtt{A})} \quad \text{for any A}$$

[&I]
$$\frac{\begin{array}{l} P \\ Q \end{array}}{P \wedge Q}$$

[RC]
$$\frac{\begin{array}{l} P \Rightarrow Q \\ Q \Rightarrow R \end{array}}{P \Rightarrow R}$$

[MG]
$$\frac{\begin{array}{l} P \Rightarrow Q \\ \neg P \Rightarrow Q \end{array}}{Q}$$

[∨D]
$$\frac{\begin{array}{l} P \vee Q \\ \neg P \end{array}}{Q}$$

[MT]
$$\frac{\begin{array}{l} P \Rightarrow Q \\ \neg Q \end{array}}{\neg P}$$

[&E]
$$\frac{P \wedge Q}{P}$$

[∨I]
$$\frac{P}{P \vee Q}$$

[⋈]
$$\frac{\begin{array}{l} P \\ \neg P \end{array}}{Contradiction}$$

. . .

Resolution Rule (PC)

- Simple example:

$\neg$Man(x)	v	Mortal(x)
Man(Socrates)		
		Mortal(Socrates)

using binding list λ ={ x/Socrates }

- Subst(Man(Socrates), λ) = Subst(Man(x), λ)
- Subst(Mortal(x), λ) = Mortal(Socrates)

- General:

$\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_n$	
	$\beta_1 \vee \beta_2 \vee \ldots \vee \beta_m$
$\alpha'_1 \vee \alpha'_2 \vee \ldots \quad \vee$	$\beta'_2 \vee \ldots \vee \beta'_m$

where there is a binding list, λ, s.t.

Subst(α_n, λ) = $\neg$Subst(β_1, λ)

Subst(α_i, λ) = α'_i $\forall$ i

Subst(β_j, λ) = β'_j $\forall$ j

+ Subsumption:
if KB includes P(x) , remove “P(A)”, “P(x) v Q(w)”
A v B, remove A v B v C

Requirement of Resolution

For Resolution to work, need:

1. Particular type of proof procedure to be "complete"

   A: Called Refutation Proof

   [… try to find contradiction…]

2. To express information as Conjunction of Disjunctions

   A: Called *Conjunctive Normal Form*

   [... is universal; can eliminate $\Rightarrow$, $\exists$, …]

3. Process that takes two literals p, q and returns binding-list λ s.t.

   Subst(p, λ) = Subst(q, λ)

   A: Called Unification

   [… is well defined, and efficient, and …]

1. Refutation Proof

- Deduction Theory

  $KB \models \sigma$ *iff*
  $KB \cup \neg\sigma$ is inconsistent *iff*
  $KB \cup \neg\sigma \models \{\}$

- To prove σ:

  Add $\neg\sigma$ to KB
  (Attempt to) Prove a contradiction

- **Resolution is Refutation Complete (in PC)**
  If $KB \models \sigma$ then
  $\exists$ resolution proof of $\{\}$ from $KB \cup \neg\sigma$

2. Conversion to Conjunctive Normal Form

0: $\forall x\ [(\forall y\ P(x, y)) \Rightarrow \neg\forall y\ Q(x, y) \Rightarrow R(x, y)]$

1: **Eliminate implication, iff, . . .**

$\forall x\ [\neg(\forall y\ P(x, y)) \vee [\neg\forall y\ [\neg Q(x, y) \vee R(x, y)]]]$

2: **Move $\neg$ inwards**

$\forall x\ [(\exists y\ \neg P(x, y)) \vee [\exists y\ Q(x, y)\ \wedge\ \neg R(x, y)]]$

3: **Standarize variables**

$\forall x\ [(\exists y\ \neg P(x, y)) \vee [\exists z\ Q(x, z)\ \wedge\ \neg R(x, z)]]$

4: **Move quantifiers left**

$\forall x\ \exists y\ \exists z\ [\neg P(x, y)) \vee [Q(x, z)\ \wedge\ \neg R(x, z)]]$

5: **Skolemize (remove existentials); Drop $\forall$s**

$\neg P(x, F_1(x)) \vee [Q(x, F_2(x))\ \wedge\ \neg R(x, F_2(x))]$

6: **Distribute $\wedge$ over $\vee$**

$[\neg P(x, F_1(x)) \vee Q(x, F_2(x))]$

$\wedge\ [\neg P(x, F_1(x)) \vee \neg R(x, F_2(x))]$

7 : **Change to SET notation**

$\left\{ \begin{array}{l} \neg P(x, F_1(x))\ \vee Q(x, F_2(x)), \\ \neg P(x, F_1(x))\ \vee\ \neg R(x, F_2(x))\ \} \end{array} \right\}$

8 : **Make variables unique**

$\left\{ \begin{array}{l} \neg P(x_1, F_1(x_1))\ \vee Q(x_1, F_2(x_1)), \\ \neg P(x_2, F_1(x_2))\ \vee\ \neg R(x_2, F_2(x_2))\ \} \end{array} \right\}$

Skolemizing

- To convert arbitrary PredicateCalculus to "Conjunctive Normal Form"
  need to eliminate ∃
- **A**: Just "name" it ... Using *new* name, to avoid conflicts
- Eg: "There is a rich person" $\exists\, x\ Rich(x)$
  becomes $Rich(\ G_1\ )$
  where G_1 is a new "Skolem constant"
- Note: $Rich(\ G_1\ )$ will NOT unify with
  $Rich(Russ)$ nor
  $Rich(\ \beta\ )$ for any β appearing in KB.
- Eg:

$$\exists k \frac{d}{dy}\left(k^{y}\right) = k^{y} \longrightarrow \frac{d}{dy}\left(e^{y}\right) = e^{y}$$

- Trickier when ∃ is inside ∀ ...

Skolemization #2

- "Everyone has a heart."
  - $\forall X$ person(X) $\Rightarrow$ $\exists Y$ heart(Y) & has(X, Y)
  - **Incorrect**: $\forall X$ person(X) $\Rightarrow$ heart($\mathbf{h_1}$) & has(X, $\mathbf{h_1}$)

    ... ?everyone has the SAME heart $\mathbf{h_1}$?
  - **Correct**:

    $\forall X$ person(X) $\Rightarrow$ heart(**h(X)**) & has(X, **h(X)**)

    where **h(.)** is a new symbol ("Skolem function")
- Skolem function arguments:

  all enclosing universally quantified variables
- Eg: $\forall A\ \forall B\ \exists C\ \forall D\ \exists E$ g(A,B, C, D, E)

  $\forall A\ \forall B\ \forall D$ g(A,B, $f_C(A,B)$, D, $f_E(A,B,D)$)

Skolemization #2

Skolemizing procedure (to remove existentials)

For each existential X,

let $Y_1, \ldots Y_m$ be the universally quantified variables that are quantified to the LEFT of X's "∃Y".

Generate new function symbol, g_X, of *m* variables.

Replace each X with $g_X(Y_1, \ldots Y_m)$.

(Write $g_X()$ as g_X.)

$$\forall Y\ \exists X\ \ \phi(X)\ \&\ \rho(Y) \rightarrow \forall Y\ \phi(\mathbf{g_X(Y)})\ \&\ \rho(Y)$$

$$\exists X\ \forall Y\ \ \phi(X)\ \&\ \rho(Y) \rightarrow \forall Y\ \phi(\mathbf{g_X})\ \&\ \rho(Y)$$

Requirement of Resolution

For Resolution to work, need:

1. Particular type of proof procedure to be “complete"

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   [… try to find contradiction…]

2. To express information as Conjunction of Disjunctions

   A: Called *Conjunctive Normal Form*

   [... is universal; can eliminate $\Rightarrow$, $\exists$, …]

3. Process that takes two literals p, q and returns binding-list λ s.t.

   Subst(p, λ) = Subst(q, λ)

   A: Called Unification

   [… is well defined, and efficient, and …]

3. Unification (Specification)

Variables capitalized

- Fancy Match
- Unify(p, q) = σ
  - p, q: atomic propositions (w/variables)
  - σ : binding list …
    - *Fail* or
    - $\{V_1/t_1, V_2/t_2, \ldots, V_n/t_n\}$ where
      - V_i's are distinct,
      - each t_j is { constant, variable, functional expr. }
      - no V_i appears in any t_j
- If non-Fail,

  Subst(p, σ) = Subst(q, σ)

  … ie, σ makes p and q look the same

Substitution

- Substitution is set $\{ V_1/t_1 \;\; V_2/t_2 \;\; \ldots \;\; V_n/t_n \}$ where
  - V_i are distinct variables
  - t_i are terms that do not use any of the V_js
- Examples:

{ X / a }

{ X / a, Y / b, Z / f(a, W) }

{ X / W, Y / f(W), Z/W }

~~{ f(X) / a }~~

~~{ X / a, X / b }~~

~~{ X / f(X) }~~

~~{ X / f(Y), Y / g(q) }~~

{ X / f(g(q)), Y / g(q) }

Applying a Substitution

- Given $\begin{cases} t & \text{— a term} \\ \sigma & \text{— a substitution} \end{cases}$

  "$t\sigma$" is the term resulting from applying substitution σ to term t.

f(a, h(Y,b), X)	{ X/b }	=	f(a, h(Y,b), b)
f(a, h(Y,b), X)	{ X/b Y/f(Z) }	=	f(a, h(f(Z),b), b)
f(a, h(Y,b), X)	{ X/Z Y/f(Z,a) }	=	f(a, h(f(Z,a),b), Z)
f(a, h(Y,b), X)	{ W/Z }	=	f(a, h(Y,b), X)

- σ need not include *all* variables in t;
  σ can include variables *not* in t

Composition of Substitutions

- Composition:

  $\sigma \circ \theta$ is composition of substitutions σ, θ

  For any term t, $t[\sigma \circ \theta\,] = (t\,\sigma\,)\,\theta$

- Example:

  $f(X)\ [\ \{X/Z\} \circ \{Z/a\}] = (\ f(X)\ \{X/Z\}\ )\ \{Z/a\}$

  $= f(Z)\ \{Z/a\}$

  $= f(a)$

- $\sigma \circ \theta$ is a substitution (usually)
- Eg:
  - $[\ \{X/a\} \circ \{Y/b\}\ ] = \{X/a, Y/b\}$
  - $[\ \{X/Z\} \circ \{Z/a\}\ ] = \{X/a, Z/a\}$

Unifiers

- t_1 and t_2 are unified by θ iff $t_1\ \theta = t_2\ \theta$

  Then θ is called a unifier

  t_1 and t_2 are unifiable

Examples:

t_1	t_2	unifer	term
f(b,c)	f(b,c)		
f(X,b)	f(a,Y)		
f(a,b)	f(c,d)		
f(a,b)	f(X,X)		
f(X,a)	f(Y,Y)		
f(g(U),d)	f(X,U)		
f(X)	f(g(X))		
f(X,g(X))	f(Y,Y)		
f(X)	f(Y)		

- Both t_1 and t_2 can have variables

Multiple Unifiers

Unifier for $t_1 = f(X)$ and $t_2 = f(Y)$

θ	$t_1\theta = t_2\theta =$
{X/Y}	f(Y)
{Y/X}	f(X)
{Y/a X/a}	f(a)
{Y/g(b,Z) X/g(b,Z)}	f(g(b(Z)))
{X/Y W/f(q,Z)}	f(Y)

- {Y/X} and {X/Y} make sense, but
  - { Y/a X/a } has irrelevant constant
  - { X/Y W/g } has irrelevant binding (W)
- Adding irrelevant bindings:

  ∞ unifiers!
- Is there a best one ?

Quest for Best Unifier

Wish list:

- No irrelevant constants
  - So {Y/X} preferred over { Y/a, X/a }
- No irrelevant bindings
  - So {Y/X} preferred over { Y/X, W/f(4,Z) }

INFERIOR unifier = composition of Good Unifier + another substitution

- Spse λ_1 has constant where λ_2 has varia

  Eg, λ_1 = { X/a, Y/a } , λ_2 = {X/Y}

  Then $\exists$ substitution μ s.t. $\lambda_2 \circ \mu = \lambda_1$

  Eg, μ = {Y/a} : {X/Y} $\circ$ {Y/a} = { X/a, Y/a }
- Spse λ_1 has extra binding over λ_2

  (Eg, λ_1 = {X/a, Y/b} , λ_2 = {X/a})

  Then $\exists$ substitution μ s.t. $\lambda_2 \circ \mu = \lambda_1$

  Eg, μ = {Y/b} : {X/a} $\circ$ {Y/b} = {X/a, Y/b}

Most General Unifier

- σ is a *mgu* for t_1 and t_2 iff
  - σ unifies t_1 and t_2, and
  - $\forall\ \mu$: unifier of t_1 and t_2,

    $\exists$ substitution, θ, s.t. $\sigma \circ \theta = \mu$.

  (Ie, for all terms t, $t\mu = (t\sigma)\theta$.)
- Example: $\sigma = \{\texttt{X/Y}\}$ is mgu for $\texttt{f(X)}$ and $\texttt{f(Y)}$

  Consider unifier $\mu = \{\texttt{X/a} \quad \texttt{Y/a}\}$.

  Use substitution $\theta = \{\texttt{Y/a}\}$:

$$\begin{aligned} \texttt{f(X)}\mu &= \texttt{f(X)}\{\texttt{X/a Y/a}\} \\ &= \texttt{f(a)} \end{aligned}$$

$$\begin{aligned} \texttt{f(X)}[\sigma \circ \theta] &= (\texttt{f(X)}\sigma) \\ &= (\texttt{f(X)}\{\texttt{X/} \\ &= \texttt{f(Y)}\{\texttt{Y/a} \\ &= \texttt{f(a)} \end{aligned}$$

(μ is NOT a mgu, as $\neg\exists\theta'$ s.t. $\mu \circ \theta' = \sigma$!)

Similarly, $\texttt{f(Y)}\mu = \texttt{f(a)} = \texttt{f(Y)}[\sigma \circ \theta]$

MGU – Example#2

- A mgu for

f(W, g(Z), Z)

f(X, Y, h(X))

is { X/W Y/g(h(W)) Z/h(W) }

MGU Procedure

```
Recursive Procedure MGU (x,y)
    If  x=y ⇒ Return ()
    If  Variable(x) ⇒ Return( MguVar(x,y) )
    If  Variable(y) ⇒ Return( MguVar(y,x) )
    If  Constant(x) or Constant(y) ⇒ Return( False )
    If  Not(Length(x) = Length(y)) ⇒ Return( False )
    g ← []
    For  i = 1..Length(x)  do
      s ← MGU( Part(x,i), Part(y,i) )
      g ← Compose(g,s)
      x ← Substitute(x,g)
      y ← Substitute(y,g)
    Return( g )
End MGU


Procedure MguVar (v,e)
    If    Includes(v,e) ⇒ Return( False )
    Else  Return( [v/e] )
End
```

MGU (con't)

- Notes:
  - If t_1 and t_2 are unifiable, then ∃ a mgu
  - Can be more than 1 mgu

    but they differ only in variable names
  - Not every unifier is a mgu.
  - A mgu uses constants only as necessary.
- Implementation: ∃ fast algorithm that computes a mgu of t_1 and t_2, if one exists;

  or reports failure.

- (Slow part is verifying legal substitution:

  none of v_i appear in any t_j.

  Avoid by resetting Prolog's *occurscheck* parameter.)

Example

- Natural Language

Jack owns a dog.
Every dog owner is an animal lover.
No animal lover kills an animal.
Either Jack or Curiosity killed the cat (named Tuna).
Did Curiosity kill the cat?

- In predicate calculus:

$\exists X$ dog(X) & owns(jack, X)
$\forall X$ ($\exists$ Y dog(Y) & owns(X, Y)) $\Rightarrow$ animalLover(X)
$\forall X$ animalLover(X) $\Rightarrow$ ($\forall$Y animal(Y) $\Rightarrow$ $\neg$kills(X, Y))
kills(jack, tuna) v kills(curiosity, tuna)
cat(tuna)
$\forall X$ cat(X) $\Rightarrow$ animal(X)
$\neg$kill(curiousity, tuna)

- Now what?

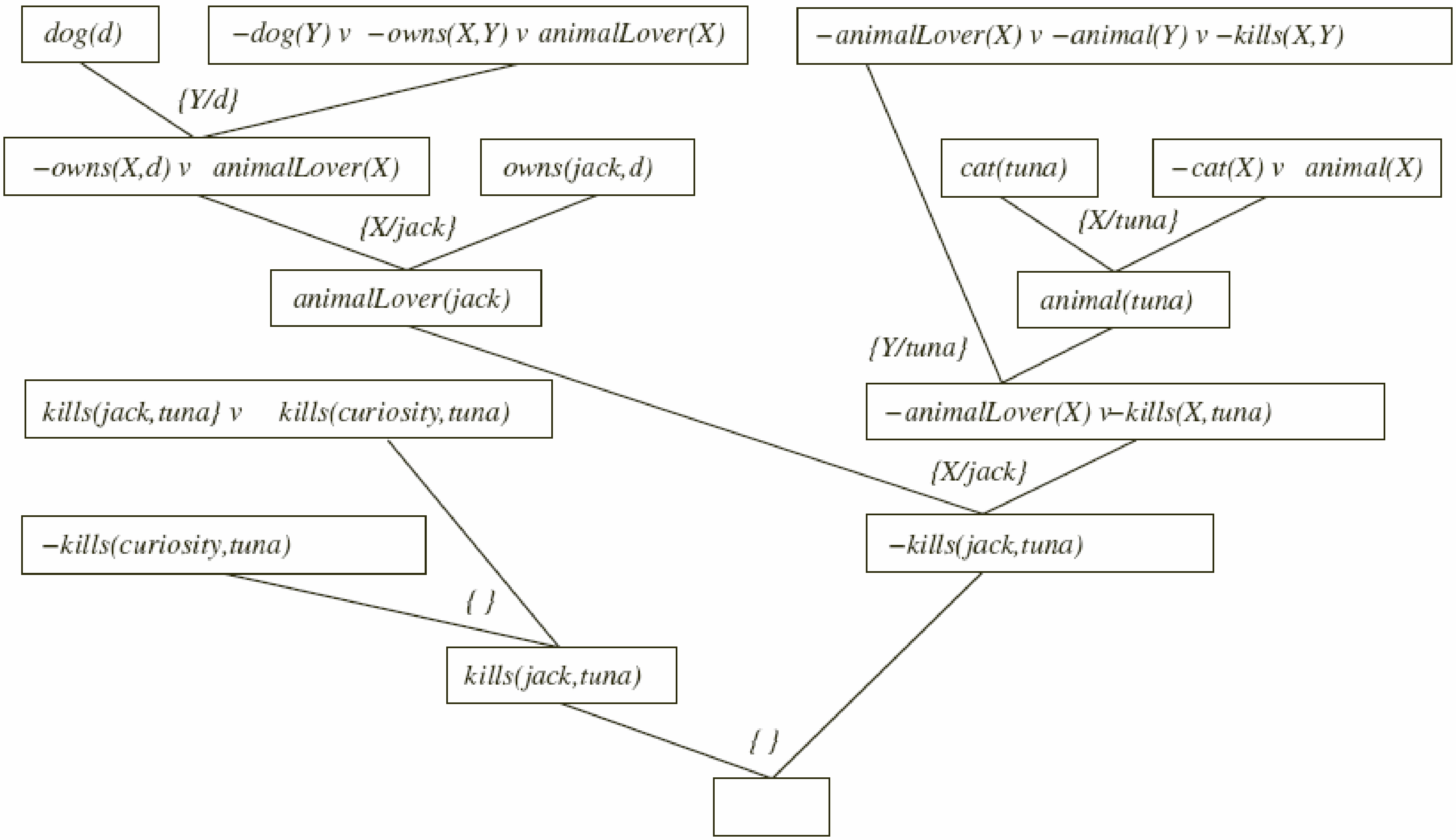
CNF Form

- ... in Conjunctive Normal Form

$\exists$X dog(X) & owns(jack, X)
$\forall$X ($\exists$ Y dog(Y) & owns(X, Y)) $\Rightarrow$ animalLover(X)
$\forall$X animalLover(X) $\Rightarrow$ ($\forall$Y animal(Y) $\Rightarrow$ $\neg$kills(X, Y))
kills(jack, tuna) v kills(curiosity, tuna)
cat(tuna)
$\forall$X cat(X) $\Rightarrow$ animal(X)
$\neg$kill(curiousity, tuna)

- Note: Uniform structure
  Use new constants / functions: d
  for existentials ("Skolemizing")
  $\Rightarrow$ easier to refer to those objects

Refutation Proof by Resolution



“Tricks”

- 1. Refutation Proof

- 2. Normalization: put in CNF form
  - Skolemize ... remove ∃

    (by giving arbitrary, but unique name to ∃ objects)
  - remove quantifiers

- 3. Unification: matching variables/ terms

  to make literals look similar

Properties of Resolution

- **Sound**
  - KB $\vdash_{RR}$ σ only if
    σ is true in EVERY world in which KB holds.
- **Refutation Complete**
  - KB $\vdash_{RR}$ σ whenever
    σ is true in EVERY world in which KB holds.
    (as $\vdash_{Resln}$ is []-complete)
- **Semi-Decidable in Predicate Calculus**
  - KB $\vdash_{RR}^{?}$ σ: If Yes, returns that answer eventually
    If No, may never return
- **Intractable**
  Exponential in |KB| for Propositional Logic
  (Linear for Proposition HORN)