RN, Chapter 7.4 - 7.8

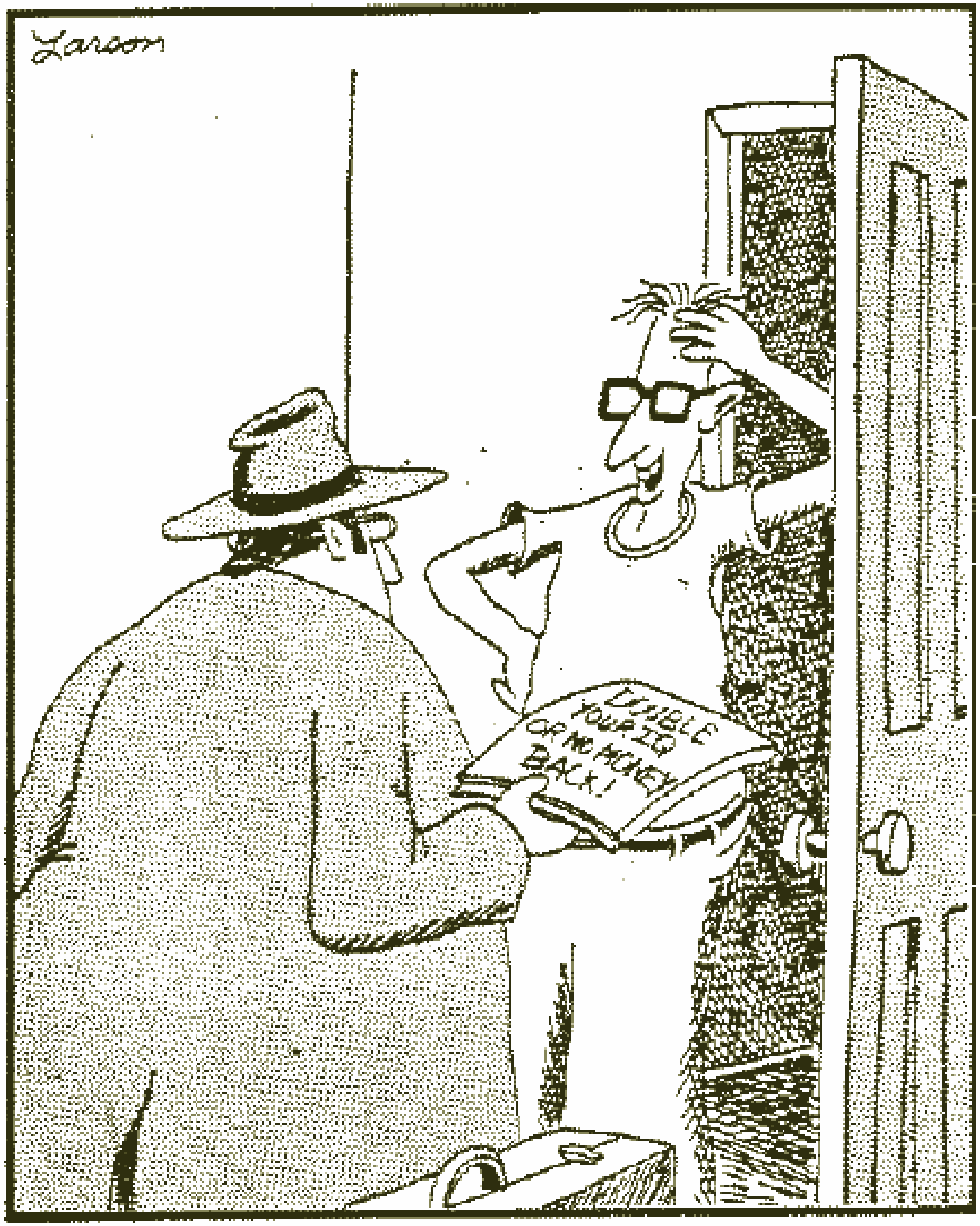
Propositional Logic

Logical Agents

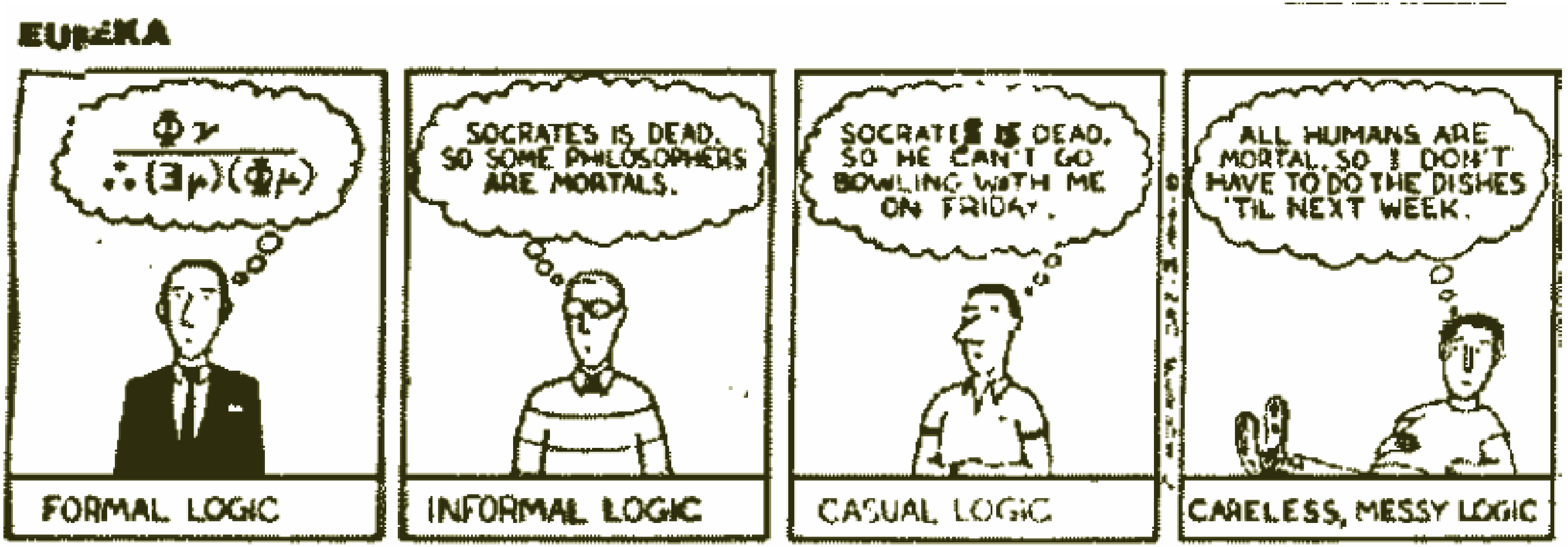
- Reasoning [Ch 7 – 7.3]
- Propositional Logic [Ch 7.4 - 7.8]
  - Syntax
  - Semantics
    - Models
    - Entailment
  - Proof Process
    - Forward / Backward chaining
    - Resolution
- Predicate Calculus
  - Representation [Ch 8]
  - Inference [Ch 9]
- Implemented Systems [Ch 10]
- Applications [Ch 8.4,10]
- Planning [Ch 11]

Logic in General

Logics are formal languages
for representing information
such that conclusions can be drawn



"Well, I dunno… Okay, sounds good to me."


EUREKA
Φγ
∴ (∃μ)(Φμ)
FORMAL LOGIC
SOCRATES IS DEAD, SO SOME PHILOSOPHERS ARE MORTALS.
INFORMAL LOGIC
SOCRATES IS DEAD, SO HE CAN'T GO BOWLING WITH ME ON FRIDAY.
CASUAL LOGIC
ALL HUMANS ARE MORTAL, SO I DON'T HAVE TO DO THE DISHES 'TIL NEXT WEEK.
CARELESS, MESSY LOGIC

Components of a Logic

- **Syntax** defines the sentences in the language
  ... what does it look like?
- **Semantics** define "meaning" of sentences;
  i.e., define truth of a sentence in a world.
  How is it linked to the world?
- **Proof Process** "new facts from old"
  find implicit information... "pushing symbols"

- Eg, wrt arithmetic
  - $x+2 \geq y$ is sentence; ~~x2+y >~~ is not
  - $x+2 \geq y$ is true iff
    the number $x+2$ is no less than the number y
  - $x+2 \geq y$ is ***true*** in a world where $x = 7; y = 1$
  - $x+2 \geq y$ is ***false*** in a world where $x = 0; y = 6$

Propositional Logic: Syntax

- Atomic Propositions… “basic statements about world”
  - $W_{3,4}$: Wumpus at location [3, 4]
  - $S_{1,1}$: Stench at location [1, 1]
  - …
- Build sentences from atomic propositions using connectives

$\wedge$	$\vee$	$\neg$	$\Rightarrow$	$\Leftrightarrow$
and	or	not	implies	equivalence
			if. . . then	(biconditional)

- Eg:

$$\neg S_{1,1} \Rightarrow \neg W_{1,1} \wedge \neg W_{1,2} \wedge \neg W_{2,1}$$
$$\neg S_{2,1} \Rightarrow \neg W_{1,1} \wedge \neg W_{2,1} \wedge \neg W_{2,2} \wedge \neg W_{3,1}$$
$$\neg S_{1,2} \Rightarrow \neg W_{1,1} \wedge \neg W_{1,2} \wedge \neg W_{2,2} \wedge \neg W_{1,3}$$
$$S_{1,2} \Rightarrow W_{1,1} \vee W_{1,2} \vee W_{2,2} \vee W_{1,3}$$

Semantics… based on Models

- "**Model**" ≡ "completely specified possible world"
  Every claim is either true or false

$m \models \varphi$ means formula φ holds in model m
Otherwise $m \not\models \varphi$

- **Propositional case:** Complete assignment
- Eg,

  $m_1 \models A$

	A	B	C	D
m_1	+	0	0	+

  "A is true in m_1" … "m_1 is a model of A"

- Also… $m_1 \models D$ $m_1 \not\models C$
- What about… $\neg B$? $A \vee B$? … $A \,\&\, \neg C \,\&\, D$?

Propositional logic: Semantics

- Each model specifies { true, false } for each proposition symbol
- Eg,

	A	B	C	D
m_1	0	+	0	+

- Rules for evaluating truth wrt model *m*:

$\neg S$	is true iff	S	is false		
$S_1 \wedge S_2$	is true iff	S_1	is true <u>and</u>	S_2	is true
$S_1 \vee S_2$	is true iff	S_1	is true <u>or</u>	S_2	is true
$S_1 \Rightarrow S_2$	is true iff	S_1	is false <u>or</u>	S_2	is true
"	is false iff	S_1	is true <u>and</u>	S_2	is false

$$
\begin{aligned}
m \models \neg S &\equiv m \not\models S \\
m \models S_1 \wedge S_2 &\equiv m \models S_1 \text{ and } m \models S_2 \\
m \models S_1 \vee S_2 &\equiv m \models S_1 \text{ or } m \models S_2 \\
m \models S_1 \Rightarrow S_2 &\equiv m \not\models S_1 \text{ or } m \models S_2
\end{aligned}
$$

Propositional logic: Semantics

	A	B	C	D
m_1	0	0	0	+

$\neg S$	is true iff	S	is false		
$S_1 \wedge S_2$	is true iff	S_1	is true and	S_2	is true
$S_1 \vee S_2$	is true iff	S_1	is true or	S_2	is true
$S_1 \Rightarrow S_2$	is true iff	S_1	is false or	S_2	is true
"	is false iff	S_1	is true and	S_2	is false

$$m \models \neg S \equiv m \not\models S$$
$$m \models S_1 \wedge S_2 \equiv m \models S_1 \text{ and } m \models S_2$$
$$m \models S_1 \vee S_2 \equiv m \models S_1 \text{ or } m \models S_2$$
$$m \models S_1 \Rightarrow S_2 \equiv m \not\models S_1 \text{ or } m \models S_2$$

$m \models^{?} A \vee (\sim B \,\&\, C)$

True if either $m \models A$ or $m \models \sim B \,\&\, C$

- But $m \not\models A$

  So need $m \models \sim B \,\&\, C$
- True if $m \models \sim B$ and $m \models C$
  - $m \models \sim B$ holds if $m \not\models B$ True…
  - So need only $m \models C$
  - Fails…

Semantics of Connectives

P	Q	¬P	P&Q	P v Q	P⇒Q	P⇔ Q
0	0	+	0	0	+	+
0	+	+	0	+	+	0
+	0	0	0	+	0	0
+	+	0	+	+	+	+

- Just need &, ¬ :
  - P v Q means ¬(¬ P & ¬ Q)
  - P ⇒ Q means ¬P v Q

    ... counterintuitive: truth value of
    "5 is even ⇒ Sam is smart" ?
- P⇔ Q means (P ⇒ Q) & (Q ⇒ P)
- "&" relatively easy, as *complete knowledge*
- "v", "¬" more difficult, as *partial information*

Models of a Formula

- Initially all 2^n models are possible

	A	B	C	D
m_1	0	0	0	0
m_2	0	0	0	+
m_3	0	0	+	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
m_{2^n-1}	+	+	+	0
m_{2^n}	+	+	+	+

- Assertion α ELIMINATES possible worlds

  Eg, $\neg A$ eliminates models m where $m \models A$

- $M(\alpha) = \{ m \mid m \models \alpha \}$ is set of all models of α
- $M(\neg A) =$

	A	B	C	D
m_1	0	0	0	0
m_2	0	0	0	+
m_3	0	0	+	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
m_{2^n-1}	0	+	+	+

Example of Entailment

- Initially:
- Background knowledge:

  Tell(KB, "$S_{12} \Rightarrow W_{11} \vee W_{13}$")

- Alive at start…

  Tell(KB, "$\neg W_{11}$")

- Smell something. . .

  Tell(KB, "S_{12}")

- Is Wumpus @ [1, 3] ?
- Is Gold @ [4, 3] ?

S_{12}	W_{11}	W_{13}	...	G_{43}
+	+	+	...	+
+	+	+	...	0
+	+	0	...	+
+	+	0	...	0
+	0	+	...	+
+	0	+	...	0
+	0	0	...	+
+	0	0	...	0
0	+	+	...	+
0	+	+	...	0
0	+	0	...	+
0	+	0	...	0
0	0	+	...	+
0	0	+	...	0
0	0	0	...	+
0	0	0	...	0

YES!

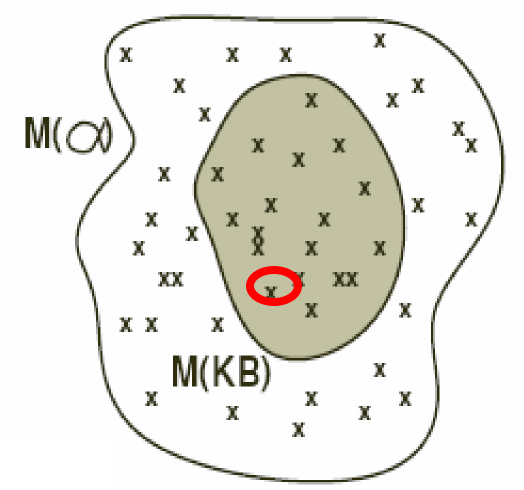
Don't know!

What to believe?

- Suppose you believe KB, and KB $\models \alpha$
  Then you should believe α !
- Why?
  1. "Believe KB"
     $\Rightarrow$ Real world m_{RW} in $\mathcal{M}$(KB)
  2. KB $\models \alpha$ means $\mathcal{M}(KB) \subseteq \mathcal{M}(\alpha)$
     $\Rightarrow$ $m_{RW} \in \mathcal{M}(\alpha)$

  . . . $m_{RW} \models \alpha$

  Ie, α holds in the Real World,
  so you should believe it!



Translate Knowledge into Action

- Include LOTS of rules like

  $A_{1,1}$ & $East_A$ & $W_{2,1} \Rightarrow \neg$Forward

- Observations re World
- Try to prove one of…

  { Forward, Turn Left, …, Shoot }

- After proof

  KB $\vdash$ Action

  perform Action

Comments on Logic

1. Why reason?
2. Entailment $\models$ vs Inference $\vdash$
3. Relation to world...
4. Succinct Representation ?

Issue#2:Entailment vs Derivation

- **Entailment** KB $\models \alpha$
  Semantic Relation:
  α MUST hold whenever KB holds

- **Derivation** KB $\vdash_i \alpha$
  Computational (Syntactic) Process:
  Maps $\langle$KB, α $\rangle$ to { Yes, No }

- $\vdash_i$ can be arbitrary but...
  want $\vdash_i$ that corresponds to $\models$!

- GOAL: $\vdash_{SC}$ that returns all+only entailments:
  For any KB, ,

  KB $\vdash_{SC} \alpha$ if-and-only-if KB $\models \alpha$

$\vdash_N$(KB, α) = No

$\vdash_A$(KB, α) =

Yes iff | α | = 1

$\vdash_{1S}$(KB, α) =

Yes iff
1-step derivation

Properties of Derivation Process

- Only 1 $\models$, but many possible proof procedures $\vdash_i$
- $\vdash_i$ is **Sound** iff

  $\vdash_i$ ONLY returns facts that must be true

  $\forall KB, \rho \quad KB \vdash_i \rho \;\Rightarrow\; KB \models \rho$

- $\vdash_i$ is **Complete** iff

  $\vdash_i$ returns every fact that must be true

  $\forall KB, \rho \quad KB \models \rho \;\Rightarrow\; KB \vdash_i \rho$

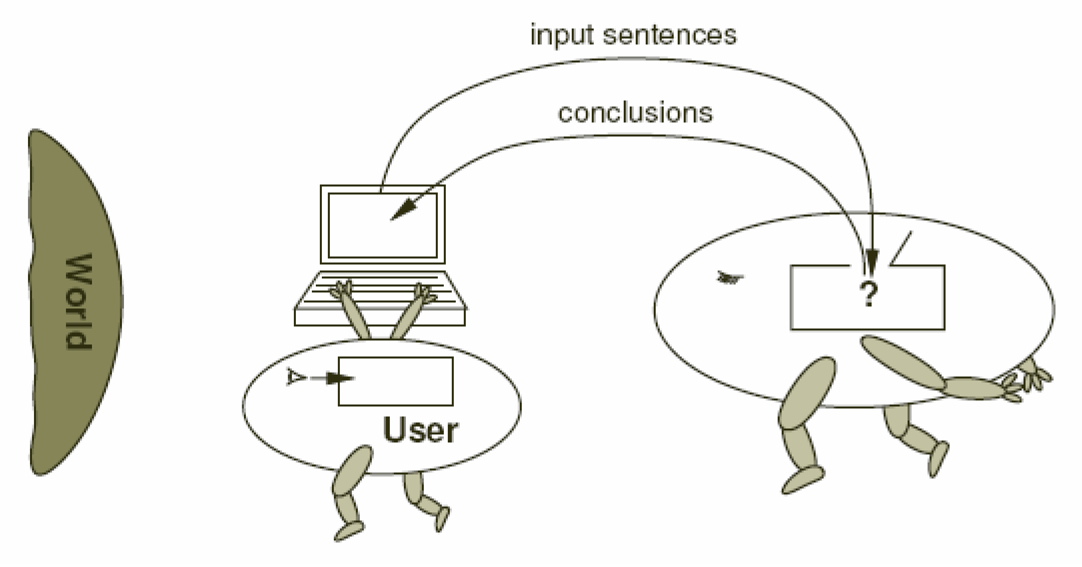
If $\vdash$ is SOUND+COMPLETE,

$\Rightarrow \quad \vdash \equiv \models$

$\Rightarrow$ Computer can IGNORE SEMANTICS and just push symbols!

Tenuous Link to Real World

- Challenge: "world" is not in computer
  . . . only a "representation" of world



- Computer only has sentences
  (hopefully about world)
  ... sensors can provide some grounding

Proof Process

- $KB = \{\varphi_j\}$ ...= SET of information "pieces"

  ... called "propositions" φ_j

  - Any rep'n will only explicitly include SOME of the true propositions

- Proof process specifies which other propositions to believe

Agent that believes KB, will also believe DERIVED propositions

  written $KB \vdash_i \rho$

  ... called "derives" (deduces, proves, ...)

- Eg:

$$\left\{ \begin{array}{l} \text{Socrates is man} \\ \text{All men are mortal} \end{array} \right\} \vdash_i \quad \text{Socrates is mortal}$$

Proof Methods

- **Model checking** … "truth table enumeration"

  (sound and complete for propositional)

  Compute complete truth table over *k* variables

  $S_{1,1}$, $S_{1,2}$, …, $W_{1,1}$, …, $B_{1,1}$, …

  Here, $\geq$ 12 variables $\Rightarrow$ $\geq 2^{12} = 4096$ rows

  Find subset where KB holds; see if α holds in all

- **Application of inference rules**

  Generate "legitimate" (sound) new sentences from old

  <u>Proof</u> = a sequence of inference rule applications

  Can use inference rules as operators in a standard search alg

Example of Model Checking

- $\alpha \equiv A \vee B$
  $KB = (A \vee C) \& (B \vee \neg C)$
- KB $\vDash^{?}$ α ?
- Check all possible models:
- KB $\vDash$ α means
  α must be true wherever KB is true

As α is True every time **KB** is true, conclude KB $\vDash$ α !

A	B	C	$A \vee C$	$B \vee \neg C$	KB	α
False	*False*	*False*	*False*	*True*	*False*	*False*
False	*False*	*True*	*True*	*False*	*False*	*False*
False	*True*	*False*	*False*	*True*	*False*	*True*
False	*True*	*True*	*True*	*True*	**True**	*True*
True	*False*	*False*	*True*	*True*	**True**	*True*
True	*False*	*True*	*True*	*False*	*False*	*True*
True	*True*	*False*	*True*	*True*	**True**	*True*
True	*True*	*True*	*True*	*True*	**True**	*True*

Challenges

- Model checking is very expensive!

  ... needs to consider 2^k models... or ∞ !
- Decision about No pit in [1, 2]

  does not depend on anything dealing at [3, 4], ...

  but $\vdash_{MC}$ still needs to consider combinatorial set of complete models

- Other inference processes can be more "local"

#2: Applying Inference Rules

- **Proof Process** is mechanic process

  Implemented by . . .

  Applying sequence of individual Inference Rules
  to initial set of propositions, to find new propositions
- Each rule is *sound*. . .

  (Ie, if believe "antecedent", must believe conclusion)
- Uses MONOTONICITY:

  If $KB1 \models \alpha$, then $KB1 \cup KB2 \models \alpha$

  Can just deal with subset of propositions

- Search issues. . .
  - which inference rule ?
  - which propositions ?

New Facts from Old: Using Inference Rules

If		“$P \Rightarrow Q$”	$\in KB$
		“P”	$\in KB$
Then	can add in	“Q” to KB	

Called “Modus Ponens”

Written:

$$[MP] \quad \frac{\begin{array}{l} P \Rightarrow Q \\ P \end{array}}{Q}$$

Verify Soundness

- Modus Ponens:

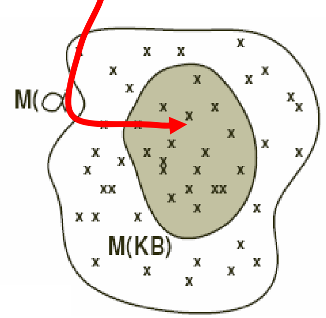
$$\frac{\alpha \Rightarrow \beta \qquad \alpha}{\beta}$$

- Truth table:

α	β	$\alpha \Rightarrow \beta$
+	+	+
~~+~~	~~0~~	~~0~~
~~0~~	~~+~~	~~+~~
~~0~~	~~0~~	~~+~~

  - Consider all worlds where $\{\alpha, \ \alpha \Rightarrow \beta\}$ hold
  - Observe: β holds here as well!

$M(\alpha, \ \alpha \Rightarrow \beta, \beta)$



$$M\begin{pmatrix} \alpha \\ \alpha \Rightarrow \beta \end{pmatrix} = M\begin{pmatrix} \alpha \\ \alpha \Rightarrow \beta \\ \beta \end{pmatrix}$$

(Sound) Inference Rules

$[MP]$ $$\frac{\alpha \Rightarrow \beta \quad \alpha}{\beta}$$

$[\&I]$ $$\frac{\alpha \quad \beta}{\alpha \wedge \beta}$$

$[RC]$ $$\frac{\alpha \Rightarrow \beta \quad \beta \Rightarrow \gamma}{\alpha \Rightarrow \gamma}$$

$[MG]$ $$\frac{\alpha \Rightarrow \beta \quad \neg\alpha \Rightarrow \beta}{\beta}$$

$[Rsln]$ $$\frac{\alpha \vee \beta \quad \gamma \vee \neg\alpha}{\gamma \vee \beta}$$

$[\vee D]$ $$\frac{\alpha \vee \beta \quad \neg\alpha}{\beta}$$

$[MT]$ $$\frac{\alpha \Rightarrow \beta \quad \neg\beta}{\neg\alpha}$$

$[\&E]$ $$\frac{\alpha \wedge \beta}{\alpha}$$

$[\vee I]$ $$\frac{\alpha}{\alpha \vee \beta}$$

. . .

Sequence of Inference Steps

1. $\alpha \,\&\, \beta$
2. $\alpha \Rightarrow \gamma$
3. $\beta \,\&\, \gamma \Rightarrow \delta$

&E 1 $\Rightarrow$

1. $\alpha \,\&\, \beta$
2. $\alpha \Rightarrow \gamma$
3. $\beta \,\&\, \gamma \Rightarrow \delta$
4. α
5. β

MP 4,2 $\Rightarrow$

1. $\alpha \,\&\, \beta$
2. $\alpha \Rightarrow \gamma$
3. $\beta \,\&\, \gamma \Rightarrow \delta$
4. α
5. β
6. γ

&I 5,6 $\Rightarrow$

1. $\alpha \,\&\, \beta$
2. $\alpha \Rightarrow \gamma$
3. $\beta \,\&\, \gamma \Rightarrow \delta$
4. α
5. β
6. γ
7. $\beta \,\&\, \gamma$

MP 7,3 $\Rightarrow$

1. $\alpha \,\&\, \beta$
2. $\alpha \Rightarrow \gamma$
3. $\beta \,\&\, \gamma \Rightarrow \delta$
4. α
5. β
6. γ
7. $\beta \,\&\, \gamma$
8. δ

Sequence of Inference Steps

M
$$\left(\begin{array}{l} \alpha \,\&\, \beta \\ \alpha \Rightarrow \gamma \\ \beta \,\&\, \gamma \Rightarrow \delta \end{array}\right)$$

Exactly the same worlds!!
So if believe FIRST,
must believe SECOND!

M
$$\left(\begin{array}{l} \alpha \,\&\, \beta \\ \alpha \Rightarrow \gamma \\ \beta \,\&\, \gamma \Rightarrow \delta \\ \alpha \\ \beta \\ \gamma \\ \beta \,\&\, \gamma \\ \delta \end{array}\right)$$

Answering Queries

- Adding Truths (Forward Chaining)

  Given KB_0, find KB_N s.t.

  $$\boxed{KB_0} \overset{ri1}{\rightsquigarrow} \boxed{KB_1} \cdots \overset{riN}{\rightsquigarrow} \boxed{KB_N}$$

  (If $\{ ri_j \}_j$ sound, then $KB_0 \models KB_N$)

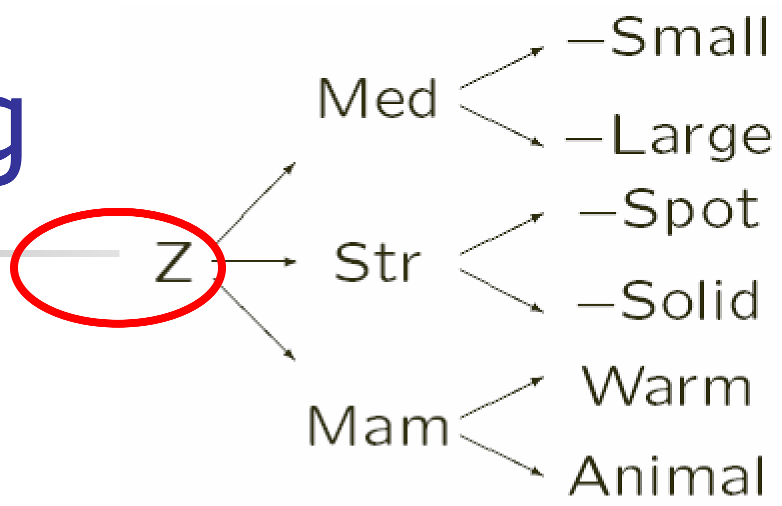
- Answering Questions (Backward Chaining)

  Given KB, σ determine if $KB_0 \models^? \sigma$

  Requires sound $\{ ri_j \}_j$ s.t.

  - $$\boxed{KB_0} \overset{ri1}{\rightsquigarrow} \boxed{KB_1} \cdots \overset{riN}{\rightsquigarrow} \boxed{KB_N}$$
  - $\sigma \in KB_N$

Forward Chaining



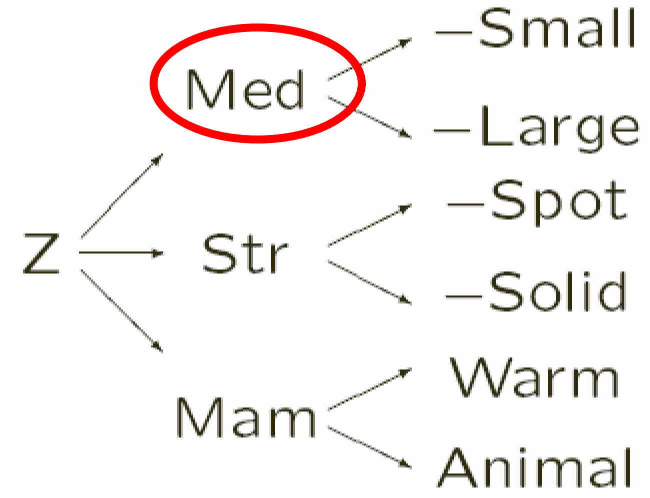
Query: Animal ?

KB_1

- Zebra
- Zebra ⇒ Medium
- Zebra ⇒ Striped
- Zebra ⇒ Mammal
- Medium ⇒ NonSmall
- Medium ⇒ NonLarge
- Striped ⇒ NonSolid
- Striped ⇒ NonSpot
- Mammal ⇒ Animal
- Mammal ⇒ Warm
- ...

KB_1 → KB_1 Med

Forward Chaining



Query: Animal ?

KB$_1$

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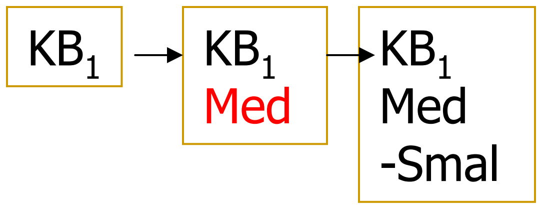
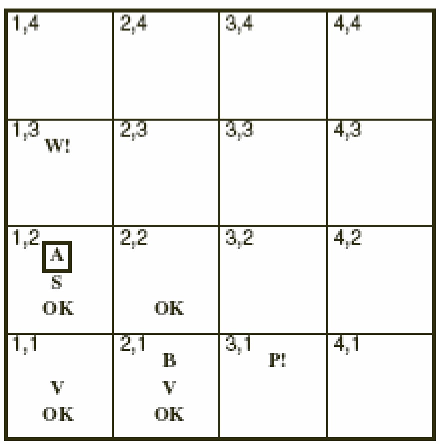
- Striped ⇒ NonSpot
- Mammal ⇒ Animal
- Mammal ⇒ Warm
- ...



Example: Is Wumpus at [1, 3] ?

$R_1: \quad \neg S_{1,1} \Rightarrow \neg W_{1,1} \wedge \neg W_{1,2} \wedge \neg W_{2,1}$
$R_2: \quad \neg S_{2,1} \Rightarrow \neg W_{1,1} \wedge \neg W_{2,1} \wedge \neg W_{2,2} \wedge \neg W_{3,1}$
$R_3: \quad S_{1,2} \Rightarrow W_{1,1} \vee W_{1,2} \vee W_{2,2} \vee W_{1,3}$

$F_1: \quad \neg S_{1,1}$
$F_2: \quad \neg S_{2,1}$
$F_3: \quad S_{1,2}$



1. MP: $\neg S_{1,1}, \quad \neg S_{1,1} \Rightarrow \neg W_{1,1} \wedge \neg W_{1,2} \wedge \neg W_{2,1}$
   $\Longrightarrow \quad \neg W_{1,1} \wedge \neg W_{1,2} \wedge \neg W_{2,1}$
2. And-Elimination: $\ldots \Longrightarrow \quad \neg W_{1,1}, \neg W_{1,2}, \neg W_{2,1}$
3. MP+AE: $\neg S_{2,1}, \quad \neg S_{2,1} \Rightarrow \neg W_{1,1} \wedge \neg W_{2,1} \wedge \neg W_{2,2} \wedge \neg W_{3,1}$
   $\Longrightarrow \quad \neg W_{1,1}, \quad \neg W_{2,1}, \quad \neg W_{2,2}, \quad \neg W_{3,1}$
4. MP: $S_{1,2}, \quad S_{1,2} \Rightarrow W_{1,1} \vee W_{1,2} \vee W_{2,2} \vee W_{1,3}$
   $\Longrightarrow \quad W_{1,1} \vee W_{1,2} \vee W_{2,2} \vee W_{1,3}$
5. Resln: $\neg W_{1,1}, \quad W_{1,1} \vee W_{1,2} \vee W_{2,2} \vee W_{1,3}$
   $\Longrightarrow \quad W_{1,2} \vee W_{2,2} \vee W_{1,3}$
6. Resln: $\neg W_{2,2}, \quad W_{1,2} \vee W_{2,2} \vee W_{1,3}$
   $\Longrightarrow \quad W_{1,2} \vee W_{1,3}$
7. Resln: $\neg W_{1,2}, \quad W_{1,2} \vee W_{1,3}$
   $\Longrightarrow \quad \boxed{W_{1,3}}$

How to Reason?

Q: Given KB, q, how to determine if $KB \models^{?} q$?

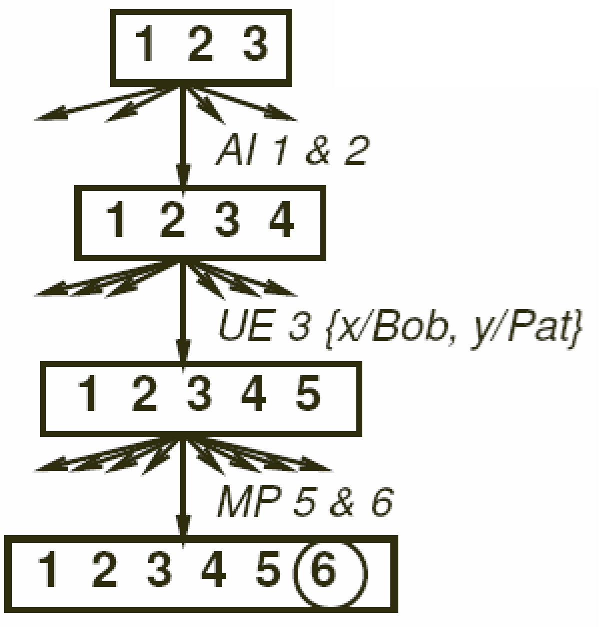
A: Select Inference Rule IR
Select fact(s) $\{F_i\}$ from KB
Apply rule IR to facts $\{F_i\}$ … to get new fact γ
. . . Add γ to KB
Repeat until find $\gamma = q$

Issues:

1. Lots of Inference Rules
   Which one to use, when?
2. Is overall system "complete"?
   If ∃ answer, guaranteed to find it?

Inference ≈ Search

- Operators ≈ inference rules
- States ≈ sets of sentences
- Goal test ≈
  does state contain query sentence?



- <u>Problem</u>:
  - huge branching factor!
  - large depth??

Resolution Rule (Propositional)

- Most Simple:

$\alpha \vee \beta$
$\neg\beta$
α

Man ∨ Mouse
¬Mouse
Man

- Almost as Simple:

Man ∨ Mouse
¬Mouse ∨ CatFood
Man ∨ CatFood

- General:

$\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_n$
$\neg\alpha_n \vee \beta_2 \vee \ldots \vee \beta_m$
$\alpha_1 \vee \alpha_2 \vee \ldots \vee \beta_2 \vee \ldots \vee \beta_m$

(View as set: So A v B v A → A v B)

Conjunctive Normal Form

- Every theory can be written in **Conjunctive Normal Form** (CNF)
  - conjunction of disjunctions of literals

clauses

(A v ¬B) & (B v ¬C v ¬D) & (B v E v ¬A)

- Can write as sets:

{ {A, ¬B}, {B, ¬C, ¬D}, {B, E, ¬A } }

- Note: {} = Falsity!

Conversion to Conjunctive Normal Form

$P \Rightarrow \neg(Q \Rightarrow R)$

- Eliminate implication, iff, ...

  $\neg P \vee \neg(\neg Q \vee R)$

- Move $\neg$ inwards

  $\neg P \vee (Q \,\&\, \neg R)$

- Distribute & over v

  $(\neg P \vee Q) \,\&\, (\neg P \vee \neg R)$

- Change to SET notation

$$\left\{ \begin{array}{l} \neg P \vee Q \\ \neg P \vee \neg R \end{array} \right\}$$

$(\alpha \Rightarrow \beta) \mapsto \neg\alpha \vee \beta$
$(\alpha \Leftrightarrow \beta) \mapsto (\neg\alpha \vee \beta)$
$\quad\quad \& \; (\alpha \vee \neg\beta)$

$\neg\neg\alpha \mapsto \alpha$
$\neg(\alpha \vee \beta) \mapsto \neg\alpha \,\&\, \neg\beta$
$\neg(\alpha \,\&\, \beta) \mapsto \neg\alpha \vee \neg\beta$

Can be EXPONENTIALLY larger than original formula

DNF $\Rightarrow$ CNF

Is Resolution Sufficient?

- Subsumes:

$$[MP] \qquad \frac{p \Rightarrow q \quad p}{q} \qquad \frac{\neg p \vee q \quad p}{q}$$

$$[MT] \qquad \frac{p \Rightarrow q \quad \neg q}{\neg p} \qquad \frac{\neg p \vee q \quad \neg q}{\neg p}$$

$$[RC] \qquad \frac{p \Rightarrow q \quad q \Rightarrow r}{p \Rightarrow r} \qquad \frac{\neg p \vee q \quad \neg q \vee r}{\neg p \vee r}$$

$$[MG] \qquad \frac{p \Rightarrow q \quad \neg p \Rightarrow q}{q} \qquad \frac{\neg p \vee q \quad p \vee q}{q}$$

$$[\bowtie] \qquad \frac{\neg p \quad p}{\{\}}$$

$$\ldots$$

- Is Resolution *sufficient*?
  Complete inference process?

Resolution $\vdash_R$ Process

Given theory KB, and query σ:

1. Find two clauses in KB

   $\alpha \equiv A \vee \mathbf{Q} \vee B$

   $\beta \equiv C \vee \neg\mathbf{Q} \vee D$

   that have complementary

   If none, return No… else…

2. Smash them!

   $\gamma \equiv A \vee B \vee C \vee D$

3. Does this new γ match σ ?
   - If so, return YES
   - If not, add to KB $\leftarrow$ KB + γ Go to 1.

Is this process COMPLETE?
Can it answer EVERY KB+query??

Resolution is NOT Complete

- Resolution $\vdash_R$ smashes together clauses

  Eg... $\{ \ldots, \alpha \vee A, \ldots, \neg A \vee \beta, \ldots\} \vdash_R \alpha \vee \beta$

- But if $KB = \{\}$, $\vdash_R$ cannot derive *anything*
- Tautologies $p \vee \neg p$ always entailed

  $$\{\} \models (p \vee \neg p)$$

  But

  $$\{\} \not\vdash_R (p \vee \neg p)$$

- Also... $\{p\} \models (p \vee p)$ but $\{p\} \not\vdash_R (p \vee p)$

  ...

Is this process COMPLETE?
Can it answer EVERY KB+query??

NO!

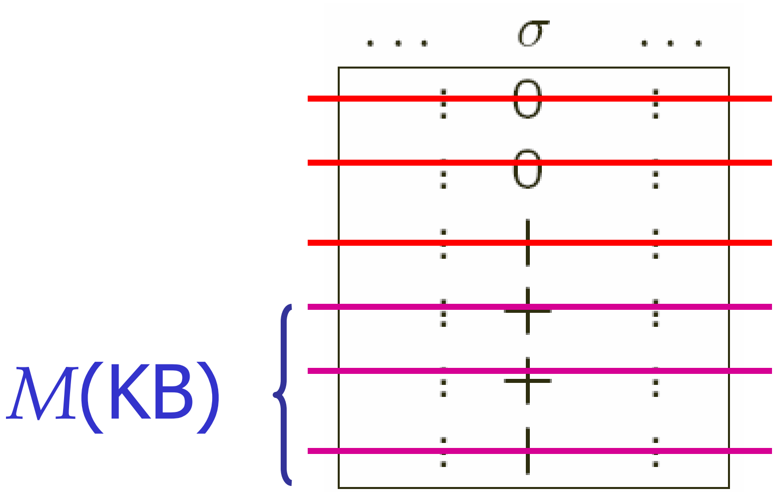
Refutation

$R \subseteq S$, then $R \cap \sim S = \{\}$

- Resolution can still be used for entailment! Using Refutation Proof :
- $KB \models \sigma$ means σ is true in all models of KB
- Now assert $\neg\sigma$
  - … ie, $KB \cup \neg\sigma$

  This removes each model where σ is true

$\Rightarrow$ it has NO models



- $M(KB \cup \neg\sigma) = \{\}$

$\Rightarrow$ $KB \cup \neg\sigma \models$ False

Refutation Proof

- Deduction Theory

$KB \models \sigma$ *iff*
$KB \cup \neg\sigma$ is inconsistent *iff*
$KB \cup \neg\sigma \models$ False

- To prove σ

Add $\neg\sigma$ to KB
If can prove a Contradiction, False, then $KB \models \sigma$

Refutation Complete

- $\vdash$ is Complete *iff*

  $\forall$ KB, σ: KB $\models \sigma \Rightarrow$ KB $\vdash \sigma$

- $\vdash$ is REFUTATION Complete *iff*

  $\forall$ KB: KB $\models \{\} \Rightarrow$ KB $\vdash \{\}$

- **Resolution $\vdash_R$ is REFUTATION COMPLETE**
- If KB $\models \sigma$ then

  $\exists$ resolution proof of False from KB $\cup \neg\sigma$

Proof…

If KB $\models \sigma$ then

$\exists$ resolution proof of False from KB $\cup \neg\sigma$

Proof: Let $RC(\Gamma)$ be deductive closure of Γ using Resl'n

Need only show: if $\{\} \notin RC(\Gamma)$,
then Γ is consistent . . . *i.e.*, Γ has model.

Build model over variables $v_1, \ldots, v_k$:
For $i = 1..k$
$\star$ if $\exists c_j \in RC(\Gamma)$ s.t.
$\neg v_i \in c_j$ and assg'n to $v_1, \ldots, v_{i-1}$ false
then $v_i \leftarrow$ false
$\star$ otherwise $v_i \leftarrow$ true

This assignment $\{\pm v_1, \ldots, \pm v_k\}$ is model for Γ !

Using Refutation Resolution

- Given KB, σ

  Let $\Gamma = KB \cup \neg\sigma$:

  Try to prove False {}, using $\vdash_R$

  $\Gamma \vdash_R^{?}$ False

  If succeed, then $KB \models \sigma$

  If fail, then $KB \not\models \sigma$

- Problem:
  - Resolution works by smashing CLAUSES!

  $\Rightarrow$ Need to encode KB, σ as clauses
- Can always be done!

Example of Resolution Process

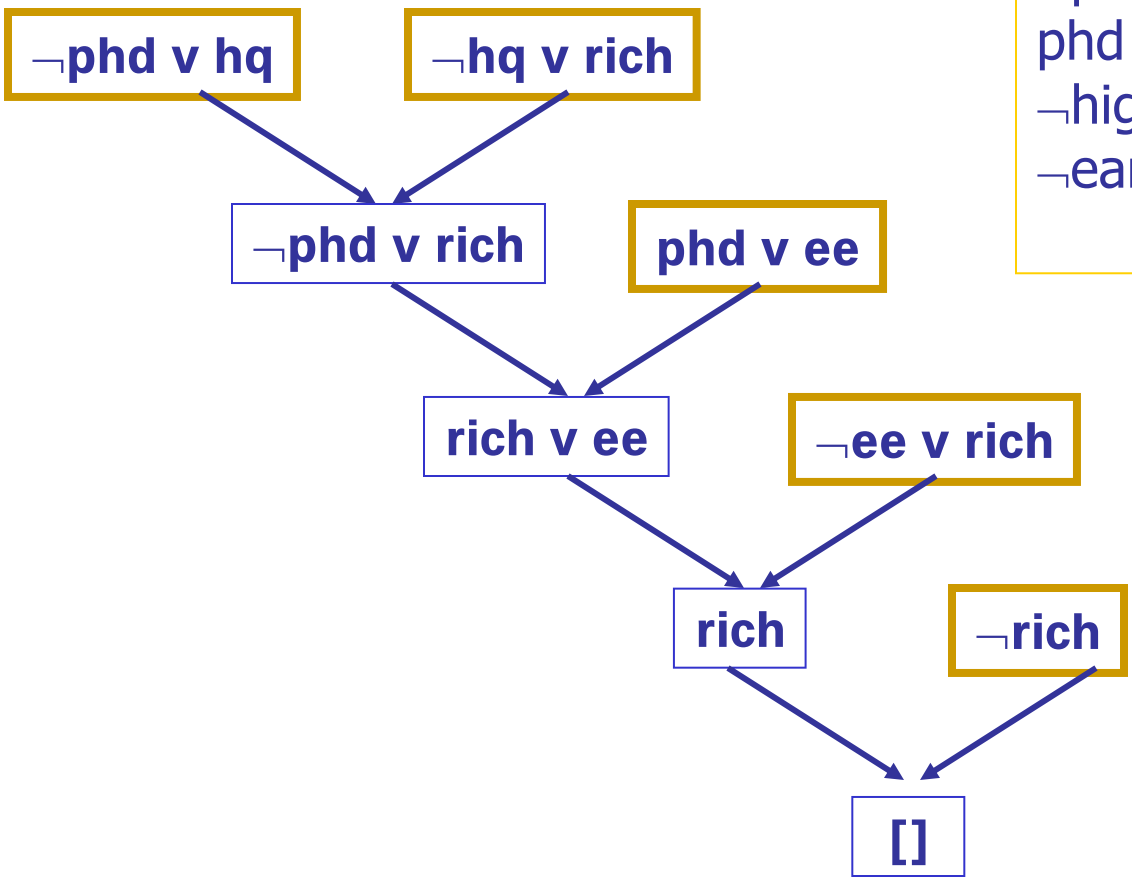
- Knowledge base

phd	$\Rightarrow$	highlyQualified
$\neg$phd	$\Rightarrow$	earlyEarnings
highlyQualified	$\Rightarrow$	rich
earlyEarnings	$\Rightarrow$	rich

- Goal: rich ?
- NOTE: simple RuleChaining will NOT work!
- Now what?
  - REFUTATION PROOF!
  - Convert to "CNF Form" (including $\neg$ of goal)
  - Resolve, seeking []
  - (Return solution)

Resolution Proof

- To prove rich from KB, add ¬rich



¬phd	v highlyQualified
phd	v earlyEarnings
¬highlyQualified	v rich
¬earlyEarnings	v rich

Inference Using Resolution

Given KB, σ

- 1. Convert KB to CNF: *CNF(KB)*
- 2. Convert $\neg\sigma$ to CNF: *CNF($\neg\sigma$)*
- 3. CNF(KB) $\cup$ CNF($\neg\sigma$) $\vdash_R$ {} ?
- If succeed, then KB $\models \sigma$
- If fail, then KB $\not\models \sigma$

- For propositional logic:
  - sound
  - complete
  - decidable
- But
  - Exponential time in general (not "just" NP-hard)
  - Linear time for Horn clauses
  - Linear time for 2-CNF clauses

Length of Resolution Proof?

- Can Resolution be FORCED to take exponentially many steps?
  - Posed [Cook / Karp, 1971/72]... related to NP vs. co-NP
  - Resolved [Haken 1985]

- Pigeon-Hole (PH) problem:

  Cannot place *n+1* pigeons in *n* holes (1/hole)
- PH takes exponentially many steps (for Resolution) no matter what order, strategy, . . .

- Important:
  - PH hidden in many practical problems
  - Makes theorem proving/ reasoning expensive
  - Contributed to recent move to model-based methods

Pigeon-Hole Principle

- $P_{i,j}$ for Pigeon i in hole j.

- Every pigeon is in some hole:

$$P_{1,1} \vee P_{1,2} \vee P_{1,3} \vee \ldots \vee P_{1,n}$$
$$P_{2,1} \vee P_{2,2} \vee P_{2,3} \vee \ldots \vee P_{2,n}$$
$$\vdots$$
$$P_{(n+1),1} \vee P_{(n+1),2} \vee P_{(n+1),3} \vee \ldots \vee P_{(n+1),n}$$

- Every pigeon is in at most one hole:

$$(\neg P_{1,1} \vee \neg P_{1,2}),\ (\neg P_{1,1} \vee \neg P_{1,3}),\ \ldots (\neg P_{1,(n-1)} \vee \neg P_{1,n})$$
$$\vdots$$
$$(\neg P_{2,1} \vee \neg P_{2,2}), \ldots, (\neg P_{2,(n-1)} \vee \neg P_{2,n})$$

- Every hole has at most one pigeon:

$$(\neg P_{1,1} \vee \neg P_{2,1}),\ (\neg P_{1,1} \vee \neg P_{3,1}),\ \ldots$$
$$(\neg P_{1,2} \vee \neg P_{2,2}),\ (\neg P_{1,2} \vee \neg P_{3,2}),\ \ldots$$
$$\vdots$$

Result

- Requires $O(n^3)$ clauses
- Resolution proof that

  PH is inconsistent

  requires dealing with at least exponential # of clauses, no matter how clauses are resolved!

  [Haken85]

$\Rightarrow$ "Method can't count"

- Can word in Predicate Calculus ... same problem

Generality; Choice Points

- As any theory can be translated to CNF
  and as resolution is []-complete,
  All deduction in terms of Resolution.
- Only decision is…
  Which (two literals in which) two clauses to (try to) Resolve?

- Eg:
  - Insist on using an atomic literal:
    Unit Resolution (F or B)
    - Only positive atomic literal: Forward reasoning
    - Only negative atomic literal: Backward reasoning
  - Set of support
  - Ancestry filtering, ordered(lock)
  - ...

Resolution Strategy I: Unit Preference

Goal: to find {} (clause w/ 0 literals)

- For $R = Resolve(P, Q)$
  $|R| = |P| + |Q| - 2$

- If |P| = 4 and |Q| = 3, then |R| = 5
  … so |R| > |P|, |Q|
  Is this progress?

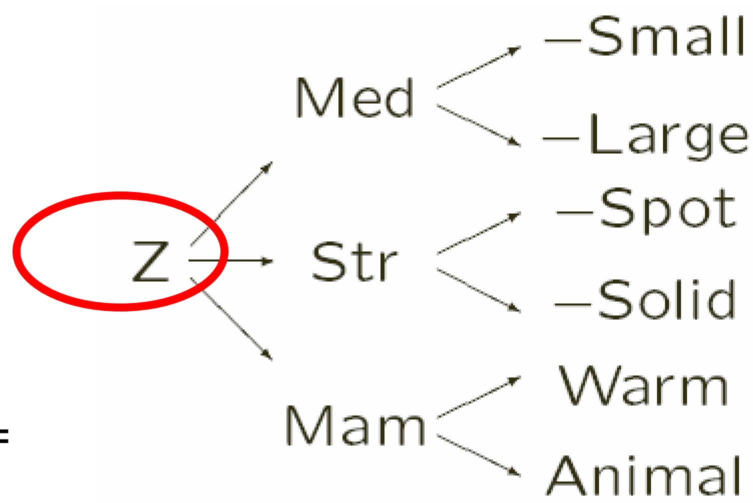
- But if |P| = 1, then |Resolve(P,Q)| = |Q| – 1
  PROGRESS towards 0 !
- Unit Preference:
  Given KB, may resolve P and Q only if
  P is single literal ("unit clause")
- Does it work?

Unit Propagation≈ Forward/Backward Reasoning

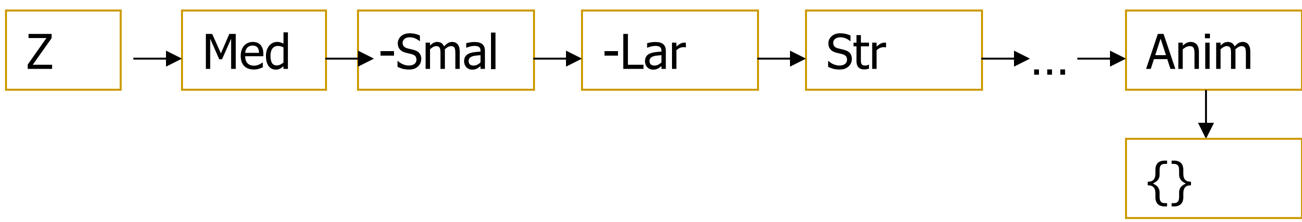
Query: Animal ?

KB$_1$

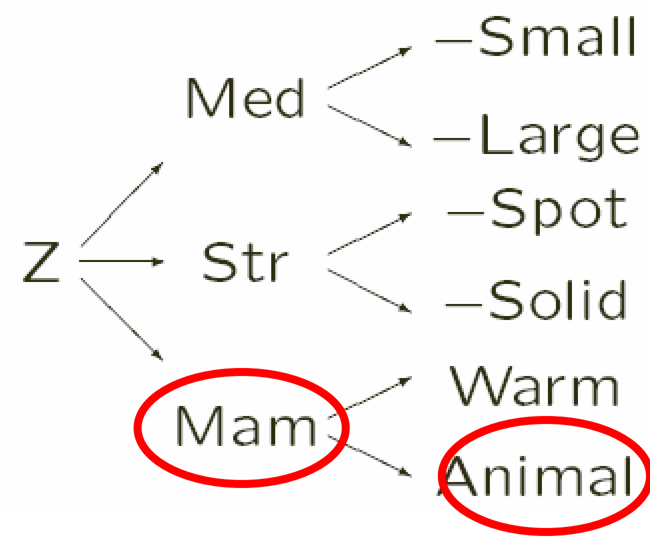
- Zebra
- ¬Zebra v Medium
- ¬Zebra v Striped
- ¬Zebra v Mammal
- ¬Medium v NonSmall
- ¬Medium v NonLarg
- ¬Striped v NonSolid
- ¬Striped v NonSpot
- ¬Mammal v Animal
- ¬Mammal v Warm
- ¬Animal



Forward Reasoning ==
use POSITIVE literal



Backward Chaining

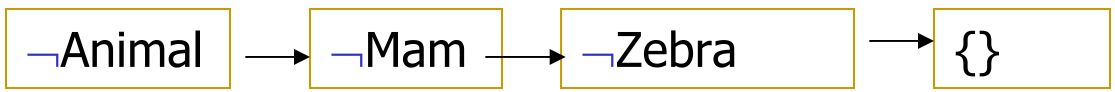


Query: Animal ?

KB$_1$

- Zebra
- ¬Zebra v Medium
- ¬Zebra v Striped
- ¬Zebra v Mammal
- ¬Medium v NonSmall
- ¬Medium v NonLarg
- ¬Striped v NonSolid
- ¬Striped v NonSpot
- ¬Mammal v Animal
- ¬Mammal v Warm
- ¬Animal

Backward Reasoning ==
use NEGATIVE literal



- **Forward chaining**:
  - Start with known facts; add in other correct facts
  - Required 9 steps
- **Backward chaining**:
  - Go from Goal to subGoal to subsubGoal to ...
  - Required 2 steps

Comparing Forward vs Backward

Query: Animal ?

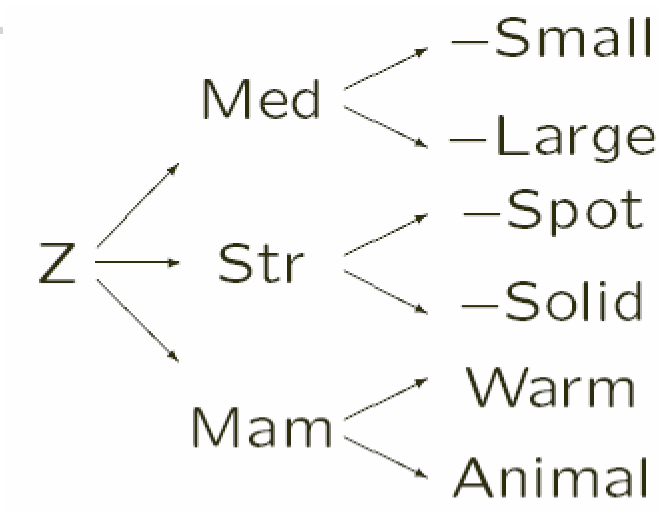
	FC	BC
KB_1	9	**2**
KB_2	**2**	8

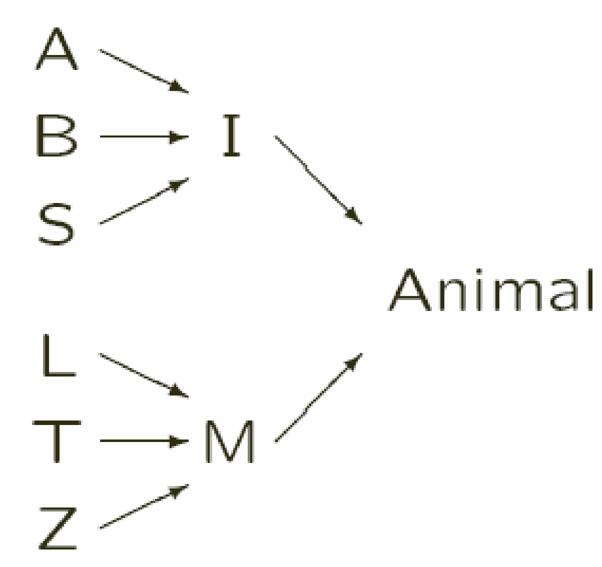
KB_1

- Zebra
- Zebra ⇒ Medium
- Zebra ⇒ Striped
- Zebra ⇒ Mammal
- Medium ⇒ NonSmall
- Medium ⇒ NonLarge
- Striped ⇒ NonSolid
- Striped ⇒ NonSpot
- Mammal ⇒ Animal
- Mammal ⇒ Warm
- ...

KB_2

- Zebra
- Ant ⇒ Insect
- Bee ⇒ Insect
- Spider ⇒ Insect
- Insect ⇒ Animal
- Lion ⇒ Mammal
- Tiger ⇒ Mammal
- Zebra ⇒ Mammal
- Mammal ⇒ Animal...





Horn Clauses … aka Rules

- Every theory can be written in **Conjunctive Normal Form** (CNF)
  - conjunction of disjunctions of literals

clauses

$$(A \vee \neg B) \; \& \; (B \vee \neg C \vee \neg D) \; \& \; (B \vee E \vee \neg A)$$

- Some theories are **Horn**
  - conjunction of Horn clauses (clauses with ≤ 1 positive literal)

$$(A \vee \neg B) \; \& \; (B \vee \neg C \vee \neg D)$$

  - Often written as set of implications:

$$(B \Rightarrow A) \; \& \; (C \; \& \; D) \Rightarrow B$$

Normal Forms

- Information needs to be in specific form to use inference rules. . .

- **Conjunctive Normal Form** (CNF - universal)
  - conjunction of disjunctions of literals

clauses

(A v ¬B) & (B v ¬C v D)

- **Disjunctive Normal Form** (DNF - universal)
  - disjunction of conjunctions of literals

terms

(A & B) v (A v ¬C) & (A & ¬D)

- **Horn Form** (restricted)
  - conjunction of Horn clauses (clauses with ≤1 positive literal)
  - Often written as set of implications:

(A v ¬B) & (B v ¬C v ¬D)

(B ⇒ A) & (C & D) ⇒ B

"Chaining" Inference Processes

- In general, need to consider $A \& B \Rightarrow C$, not just $A \Rightarrow C$
- Here, both F- and B- reasoning were also
  Unit Preferences
  Typical, but can be generalized …

- **If Horn theory:**
  - **Forward Chaining** …
    is COMPLETE for ATOMIC Queries
    DataDriven – could be done by *Tell*
  - **Backward Chaining**
    is COMPLETE for ATOMIC Queries

  - Each is worst-case *O(n)…*
    but different actual run-times ...depends on Branching Factor...

- But NOT everything is HORN!
  - $S_{1,2} \Rightarrow W_{1,1} \vee W_{1,2} \vee W_{2,2} \vee W_{1,3}$
  - $\neg S_{1,2} \vee W_{1,1} \vee W_{1,2} \vee W_{2,2} \vee W_{1,3}$

Unit Resolution

Can resolve P and Q only if . . .

- **Unit Preference**: |P| = 1

  STATUS: Not complete

  $$\left\{ \begin{matrix} A \vee B & \neg A \vee B \\ A \vee \neg B & \neg A \vee \neg B \end{matrix} \right\}$$

  But ... *Refutation Complete* for Horn clauses.

- Horn , each clause has ≤ 1 positive literal

  Horn: A A v ¬B ¬B ¬A v ¬B …

  NotHorn: A v B A v ¬Q v W

Ordered Resolution

Ordered Resolution: $\left\{ \begin{matrix} A \vee B & \neg A \vee B \\ A \vee \neg B & \neg A \vee \neg B \end{matrix} \right\}$

- Literals in each clause are ordered:

  $P = \langle p_1 \vee p_2 \vee \ldots \rangle, \quad Q = \langle q_1 \vee q_2 \vee \ldots \rangle$
- Can resolve P and Q only if p_1 corresponds to $\neg q_1$

STATUS: Refutation complete for Horn

Resolution Strategies, II

- **Set of Support**: Resolve P, Q only if P $\in$ S where S KB is "set of support".
  . . . then add resolvent to S.

  Complete if *Consistent(KB- S)*
- **Backward Reasoning**:
  Initial Support: S = negated query $\neg\sigma$
- **Forward Reasoning**:
  Initial Support: S = original KB

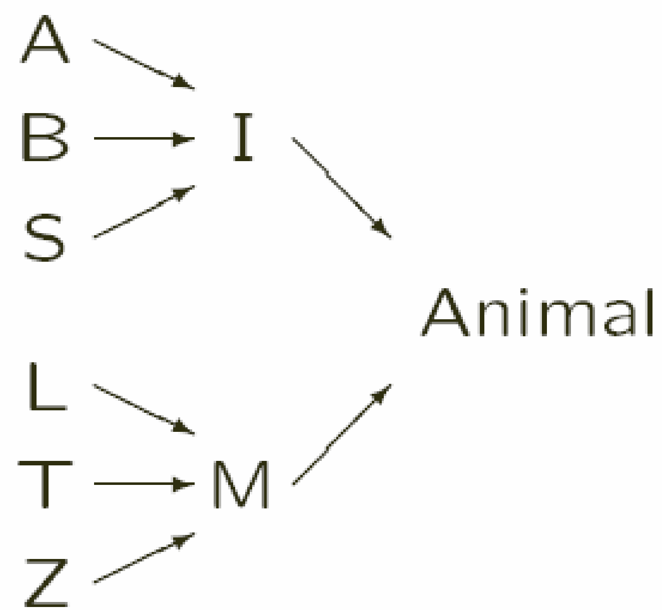
- Q: Which is better?
- A: Depends on branching factor!

Set-of-Support: Backward Reasoning

Zebra
¬Ant v Insect
¬Bee v Insect
¬Spider v Insect
¬Insect v Animal
¬Lion v Mammal
¬Tiger v Mammal
¬Zebra v Mammal
¬Mammal v Animal

¬Animal

Set of Support

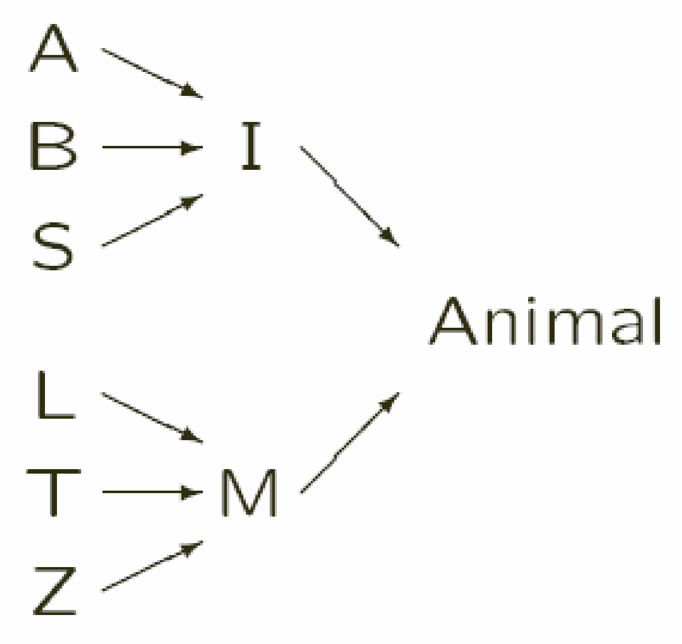


Set-of-Support: Forward Reasoning

Set of Support

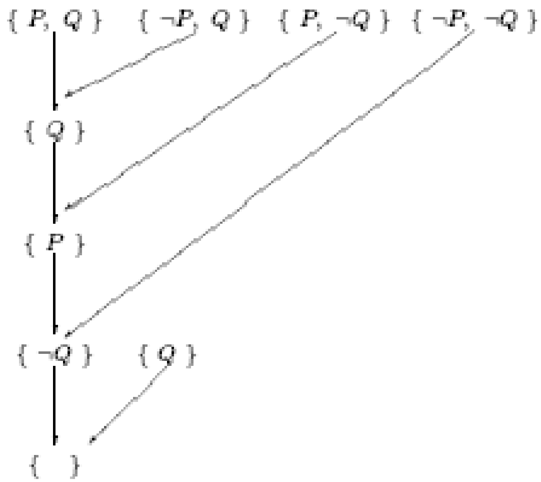
¬Animal

Zebra
¬Ant v Insect
¬Bee v Insect
¬Spider v Insect
¬Insect v Animal
¬Lion v Mammal
¬Tiger v Mammal
¬Zebra v Mammal
¬Mammal v Animal



Resolution Strategies, III

- **Input Resolution:** only if P in original KB
  STATUS: Not complete.
- **Linear resolution**: only if P in original KB
  or P is ancestor of Q in proof tree



- **STATUS**: Refutation complete
  (if KB consistent, then KB u ~σ inconsistent
  iff LinRes, starting with σ, reaches {})

Deduction Theorem, Validity, Satisfiability

- Sentence is valid iff true in all models

  Eg, $A \vee \sim A$, $A \Rightarrow A$, $(A \,\&\, (A \Rightarrow B)) \Rightarrow B$
  - … related to inference via **Deduction Theorem**:

$$KB \models \alpha \quad \text{iff} \quad (KB \Rightarrow \alpha) \text{ is valid}$$

- Sentence is satisfiable iff true in <u>some</u> model

  Eg, $A \vee B$, C
- Sentence is unsatisfiable iff true in <u>no</u> models

  Eg, $A \,\&\, \neg A$
- Satisfiability related to inference via …

$$KB \models \alpha \quad \text{iff} \quad (KB \,\&\, \neg\alpha) \text{ is unsatisfiable}$$

…. prove α by reductio ad absurdum

Basic concepts of logic

- syntax: formal structure of sentences
- semantics: truth of sentences wrt models
- entailment: necessary truth of one sentence given another
- derivation: deriving sentences from other sentences
- soundness: derivations produce only entailed sentences
- completeness: derivations can produce all entailed sentences

Summary

- Logical agents make inferences from a knowledge base

  to derive new information (used to make decisions)
- Even simple tasks (Wumpus World) require ability to
  - represent *partial* and *negated* information,
  - reason by cases, etc.
- Propositional logic often sufficient
- Resolution is sound + complete … if exponential time