Entailment

- Suppose we assert $\gamma_1$, $\gamma_2$, ...
  
  $\text{on}(a,b)$, $\text{on}(a,b) \Rightarrow \text{above}(a,b)$, ...

Then... you should believe $\gamma_1$, $\gamma_2$, ...

What else should we believe?
  
  (Should we believe a given $\beta$ – eg, “above($a,b$)”?)

- **Entailment** addresses this:
  
  You should believe all (and only) entailments.

Later: Present ALGORITHM for
  
  *computing* these entailments...
Semantics... based on Models

“Model” ≡ “completely specified possible world”

Every claim is either true or false

\[ m \models \varphi \] means

formula \( \varphi \) holds in model \( m \)

Otherwise \( m \not\models \varphi \)

Propositional case: Complete assignment

\[
\begin{array}{c|cccc}
\text{Eg,} & A & B & C & D \\
m_1 & + & 0 & 0 & + \\
\end{array}
\]

\[ m_1 \models A \]

“A is true in \( m_1 \)”

“\( m_1 \) is a model of \( A \)”

Also...

\[ m_1 \models \neg B \quad m_1 \models \neg C \quad m_1 \models D \]

• What about

\[ A \lor B \quad \ldots \quad A \land \neg C \land D \quad ? \]
Propositional logic: Semantics

- Each model specifies \{ true, false \} for each proposition symbol

Eg, $$
\begin{array}{c|cccc}
  m_1 & A & B & C & D \\
  \hline
  + & 0 & 0 & + \\
\end{array}
$$

- Rules for evaluating truth with respect to a model \( m \):

$$
\begin{align*}
  \neg S & \text{ is true iff } S \text{ is false} \\
  S_1 \land S_2 & \text{ is true iff } S_1 \text{ is true and } S_2 \text{ is true} \\
  S_1 \lor S_2 & \text{ is true iff } S_1 \text{ is true or } S_2 \text{ is true} \\
  S_1 \Rightarrow S_2 & \text{ is true iff } S_1 \text{ is false or } S_2 \text{ is true} \\
  " & \text{ is false iff } S_1 \text{ is true and } S_2 \text{ is false}
\end{align*}
$$

$$
\begin{align*}
m \models \neg S & \equiv m \not\models S \\
m \models S_1 \land S_2 & \equiv m \models S_1 \text{ and } m \models S_2 \\
m \models S_1 \lor S_2 & \equiv m \models S_1 \text{ or } m \models S_2 \\
m \models S_1 \Rightarrow S_2 & \equiv m \not\models S_1 \text{ or } m \models S_2
\end{align*}
$$
Semantics of Connectives

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$\neg P$</th>
<th>$P &amp; Q$</th>
<th>$P \lor Q$</th>
<th>$P \Rightarrow Q$</th>
<th>$P \Leftrightarrow Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>+</td>
<td>0</td>
<td>0</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>+</td>
<td>0</td>
<td>+</td>
<td>+</td>
<td>0</td>
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<tr>
<td>1</td>
<td>0</td>
<td>0</td>
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<td>1</td>
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<td>0</td>
<td>+</td>
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</tr>
</tbody>
</table>

Just need $\land$, $\neg$:

- $P \lor Q$ means $\neg(\neg P \land \neg Q)$
- $P \Rightarrow Q$ means $\neg P \lor Q$

...counterintuitive:

What’s truth value of “5 is even $\Rightarrow$ Sam is smart”? 

- $P \Leftrightarrow Q$ means $(P \Rightarrow Q) \land (Q \Rightarrow P)$

- “$\land$” – relatively easy
  complete knowledge...

- “$\lor$”, “$\neg$” – more difficult
  partial information
Interpretation

- **Vocabulary: (literals)**
  
  \[ \text{on}_{ab}: \text{ if } a \text{ is immediately on } b \]
  
  \[ \text{on}_{bc}: \text{ if } b \text{ is immediately on } c \]
  
  \[ \ldots \]
  
  \[ \text{above}_{ab}: \text{ if } a \text{ is (somewhere) above } b \]
  
  \[ \text{above}_{ac}: \text{ if } a \text{ is (somewhere) above } c \]

- **An “interpretation” (aka “world”)**
  assigns True xor False to each literal

\[ n \text{ literals } \Rightarrow 2^n \text{ possible worlds, } \mathcal{M} \]

<table>
<thead>
<tr>
<th></th>
<th>on_{ab}</th>
<th>on_{bc}</th>
<th>\ldots</th>
<th>above_{ab}</th>
<th>above_{ac}</th>
<th>\ldots</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m_1 )</td>
<td>+</td>
<td>+</td>
<td>\ldots</td>
<td>+</td>
<td>+</td>
<td>\ldots</td>
</tr>
<tr>
<td>( m_2 )</td>
<td>+</td>
<td>+</td>
<td>\ldots</td>
<td>+</td>
<td>0</td>
<td>\ldots</td>
</tr>
<tr>
<td>( m_3 )</td>
<td>+</td>
<td>+</td>
<td>\ldots</td>
<td>0</td>
<td>+</td>
<td>\ldots</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>( m_m )</td>
<td>+</td>
<td>0</td>
<td>\ldots</td>
<td>+</td>
<td>0</td>
<td>\ldots</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>( m_{2^n} )</td>
<td>0</td>
<td>0</td>
<td>\ldots</td>
<td>0</td>
<td>0</td>
<td>\ldots</td>
</tr>
</tbody>
</table>
Models of a Formula

- Initially all $2^n$ models are possible

<table>
<thead>
<tr>
<th></th>
<th>$on_{ab}$</th>
<th>$on_{bc}$</th>
<th>$above_{ab}$</th>
<th>$above_{bc}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$m_2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>+</td>
</tr>
<tr>
<td>$m_3$</td>
<td>0</td>
<td>0</td>
<td>+</td>
<td>0</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$m_{2^n-1}$</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>0</td>
</tr>
<tr>
<td>$m_{2^n}$</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
</tbody>
</table>

- Assertion $\alpha$ ELIMINATES possible worlds
  Eg, $\neg on_{ab}$ eliminates models $m$ where $\models_m on_{ab}$

- $M(\alpha) = \{ m \mid \models_m \alpha \}$
  is set of all models of $\alpha$

$M(\neg on_{ab}) = $

<table>
<thead>
<tr>
<th></th>
<th>$on_{ab}$</th>
<th>$on_{bc}$</th>
<th>$above_{ab}$</th>
<th>$above_{bc}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$m_2$</td>
<td>0</td>
<td>0</td>
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<td>+</td>
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<tr>
<td>$m_3$</td>
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<td>+</td>
<td>0</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$m_{2^n-1}$</td>
<td>0</td>
<td>$+$</td>
<td>+</td>
<td>+</td>
</tr>
</tbody>
</table>

Logics
Dealing with Boolean Combinations

<table>
<thead>
<tr>
<th></th>
<th>on\textsubscript{ab}</th>
<th>on\textsubscript{bc}</th>
<th>above\textsubscript{ab}</th>
<th>above\textsubscript{ac}</th>
<th>⋮</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_1$</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>⋮</td>
</tr>
<tr>
<td>$m_3$</td>
<td>+</td>
<td>+</td>
<td>0</td>
<td>+</td>
<td>⋮</td>
</tr>
<tr>
<td>$m_5$</td>
<td>+</td>
<td>0</td>
<td>+</td>
<td>+</td>
<td>⋮</td>
</tr>
<tr>
<td>$m_7$</td>
<td>+</td>
<td>0</td>
<td>0</td>
<td>+</td>
<td>⋮</td>
</tr>
<tr>
<td>$m_9$</td>
<td>0</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>⋮</td>
</tr>
<tr>
<td>$m_{11}$</td>
<td>0</td>
<td>+</td>
<td>0</td>
<td>+</td>
<td>⋮</td>
</tr>
<tr>
<td>$m_{13}$</td>
<td>0</td>
<td>0</td>
<td>+</td>
<td>+</td>
<td>⋮</td>
</tr>
<tr>
<td>$m_{15}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>+</td>
<td>⋮</td>
</tr>
<tr>
<td>⋮</td>
<td>⋮</td>
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<td>⋮</td>
<td>⋮</td>
</tr>
</tbody>
</table>

$\mathcal{M}(\chi) = \{ m \mid \models m \chi \}$

- $\mathcal{M}(\text{on}_{ab}) = \{ m_1, \ldots, m_8 \}$
- $\mathcal{M}(\neg \text{on}_{ab}) = \{ m_9, \ldots, m_{16} \}$
- $\mathcal{M}(\text{on}_{ab} \land \text{on}_{ac})$
  
  $= \{ m_1, \ldots, m_8 \} \cap \{ m_1, \ldots, m_4, m_9, \ldots, m_{12} \}$
  
  $= \{ m_1, \ldots, m_4 \}$
- $\mathcal{M}(\neg \text{on}_{ab} \lor \text{above}_{ab})$
  
  $= \{ m_9, \ldots, m_{16} \} \cup \{ m_1, m_2, m_5, m_6, m_9, m_{10}, m_{13}, m_{14} \}$
  
  $= \{ m_1, m_2, m_5, m_6, m_9, \ldots, m_{16} \}$

Note: $\mathcal{M}(\neg \chi) = \mathcal{M} - \mathcal{M}(\chi)$

$\mathcal{M}(\chi \lor \xi) = \mathcal{M}(\chi) \cup \mathcal{M}(\xi)$

$\mathcal{M}(\chi \land \xi) = \mathcal{M}(\chi) \cap \mathcal{M}(\xi)$
What else to believe?

\[
\begin{array}{|c|c|c|c|c|}
\hline
& \text{on}_{ab} & \text{on}_{bc} & \text{above}_{ab} & \text{above}_{ac} \\
\hline
m_1 & + & + & + & + & \ldots \\
\hline
m_2 & + & + & + & 0 & \ldots \\
\hline
m_3 & + & + & 0 & + & \ldots \\
\hline
m_4 & + & + & 0 & 0 & \ldots \\
\hline
m_5 & + & 0 & + & + & \ldots \\
\hline
m_6 & + & 0 & + & 0 & \ldots \\
\hline
m_7 & + & 0 & 0 & + & \ldots \\
\hline
m_8 & + & 0 & 0 & 0 & \ldots \\
\hline
m_9 & 0 & + & + & + & \ldots \\
\hline
m_{10} & 0 & + & + & 0 & \ldots \\
\hline
m_{11} & 0 & + & 0 & + & \ldots \\
\hline
m_{12} & 0 & + & 0 & 0 & \ldots \\
\hline
m_{13} & 0 & 0 & + & + & \ldots \\
\hline
m_{14} & 0 & 0 & + & 0 & \ldots \\
\hline
m_{15} & 0 & 0 & 0 & + & \ldots \\
\hline
m_{16} & 0 & 0 & 0 & 0 & \ldots \\
\hline
\end{array}
\]

- Assert $\neg \text{on}_{ab} \lor \text{above}_{ab}$
  \[
  \text{RealWorld} \in \mathcal{M}(\neg \text{on}_{ab} \lor \text{above}_{ab}) = \{m_1, m_2, m_5, m_6, m_9, \ldots m_{16}\}
  \]

- Also... assert $\text{on}_{ab}$
  Hence, $\text{RealWorld in } \mathcal{M}(\neg \text{on}_{ab} \lor \text{above}_{ab}) \cap \mathcal{M}(\text{on}_{ab}) = \{m_1, m_2, m_5, m_6\}$

Note: In each such world, $\text{above}_{ab}$ also holds!

\[
\mathcal{M}(\neg \text{on}_{ab} \lor \text{above}_{ab} \land \text{on}_{ab}) \subset \mathcal{M}(\text{above}_{ab})
\]

$\Rightarrow$ We should believe $\text{above}_{ab}$!
Meaning of Entailment

- $\alpha$ is entailment of $KB$ \hspace{1cm} KB \models \alpha

  * $\alpha$ is true in all worlds where $KB$ is true
  * All models of $KB$ are models of $\alpha$
  * $\mathcal{M}(KB) \subseteq \mathcal{M}(\alpha)$

- (As we believe everything in $KB$, . . .)

  Real world is a model of $KB$
  \[ m_{RW} \in \mathcal{M}(KB) \]

- As \hspace{1cm} $\mathcal{M}(KB) \subseteq \mathcal{M}(\alpha)$

  \[ \Rightarrow m_{RW} \in \mathcal{M}(\alpha) \]
  real-world is model of $\alpha$

  ie, $\alpha$ must hold!

Possible Worlds

\[ \mathcal{M}(KB_0) \]
\[ \mathcal{M}(KB_1) \]
What to believe?

- Suppose you believe $KB$
  
  and $KB \models \alpha$

  Then you should believe $\alpha$!

Why?

1. “Believe $KB$
   
   $\Rightarrow$ Real world $m_{RW}$ in $\mathcal{M}(KB)$

2. $KB \models \alpha$ means $\mathcal{M}(KB) \subseteq \mathcal{M}(\alpha)$

   $\Rightarrow m_{RW} \in \mathcal{M}(\alpha)$

   $\ldots m_{RW} \models \alpha$

Ie, $\alpha$ holds in the Real World, so you should believe it!
**Entailment vs Derivation**

- **Entailment** \( KB \models \alpha \)
  
  **Semantic Relation:**
  
  \( \alpha \) MUST hold whenever \( KB \) holds.

- **Derivation** \( KB \vdash_i \alpha \)
  
  **Computational (Syntactic) Process:**
  
  Maps \( \langle KB, \alpha \rangle \) to \{Yes, No\}

- \( \vdash_i \) can be arbitrary but...
  
  want \( \vdash \) to correspond to \( \models \)!

**GOAL:** \( \vdash_{SC} \) which returns only entailments:

For any \( KB, \alpha \),

\[ KB \vdash_{SC} \alpha \text{ if-and-only-if } KB \models \alpha \]
Answering Queries
1. Semantical Approach

- Entailment specifies what we should believe.

- Possible procedure for deciding whether
  \[ \{ \gamma_i \} \models \beta \]

**MODEL CHECKING**
Given \( n \) literals, write \( n \times 2^n \) table.
Let \( W \) be set of all \( 2^n \) rows
For each assertion \( \gamma_i \),
  let \( W := W \cap M(\gamma_i) \)
  (\textit{ie, eliminate every row that does not satisfy } \gamma_i \text{)}
Check \( \beta \) column.
If “+” in each row (of \( W \))
  then Answer “YES: \( \{ \gamma_i \} \models \beta \)”
  else Answer “NO: \( \{ \gamma_i \} \not\models \beta \)”

- Problem: HUGE table!
  \((\infty \text{ in predicate calculus})\)

- There is another approach:
Answering Queries
2. Syntactic Approach

• Proof Process (aka “derivation”, “deduction”) is mechanic process

... for deciding whether conclusion (query) follows from premises (initial knowledge base)

First consider “forward chaining”

\[ KB \implies KB' \]

Implemented by ...
Applying sequence of individual (sound) Inference Rules to initial set of propositions, to find new ones

| Sound | \iff preserves truth |

If believe “antecedent”, must believe conclusion

• Inference Rules ≠ HornClause
Sequence of Inference Rules

1. \( \alpha \land \beta \)
2. \( \alpha \Rightarrow \gamma \)
3. \( \beta \land \gamma \Rightarrow \delta \)

&\text{E} \quad \frac{1}{1}

1. \( \alpha \land \beta \)
2. \( \alpha \Rightarrow \gamma \)
3. \( \beta \land \gamma \Rightarrow \delta \)
4. \( \alpha \)
5. \( \beta \)

MP \quad \frac{4,2}{4,2}

1. \( \alpha \land \beta \)
2. \( \alpha \Rightarrow \gamma \)
3. \( \beta \land \gamma \Rightarrow \delta \)
4. \( \alpha \)
5. \( \beta \)
6. \( \gamma \)

&\text{I} \quad \frac{5,6}{5,6}

1. \( \alpha \land \beta \)
2. \( \alpha \Rightarrow \gamma \)
3. \( \beta \land \gamma \Rightarrow \delta \)
4. \( \alpha \)
5. \( \beta \)
6. \( \gamma \)
7. \( \beta \land \gamma \)

MP \quad \frac{7,3}{7,3}

1. \( \alpha \land \beta \)
2. \( \alpha \Rightarrow \gamma \)
3. \( \beta \land \gamma \Rightarrow \delta \)
4. \( \alpha \)
5. \( \beta \)
6. \( \gamma \)
7. \( \beta \land \gamma \)
8. \( \delta \)
New Facts from Old: A Sound Inference Rule

If

"P ⇒ Q" ∈ KB
"P" ∈ KB

Then

can add "Q" to KB

Called “Modus Ponens”

Written:

\[
[MP] \quad P ⇒ Q
\]

\[
\begin{array}{c}
P \\
\hline
Q
\end{array}
\]
Verify Soundness

- Modus Ponens:

\[
\begin{array}{c}
\alpha \Rightarrow \beta \\
\alpha \\
\hline
\beta \\
\end{array}
\]

Truth table:

<table>
<thead>
<tr>
<th>\alpha</th>
<th>\beta</th>
<th>\alpha \Rightarrow \beta</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

Consider all worlds where \( \left\{ \begin{array}{c}
\alpha \\
\alpha \Rightarrow \beta \\
\end{array} \right. \) both hold.

Observe: \( \beta \) holds here as well!

- And-Introduction \( [\&I] \)

\[
\begin{array}{c}
\alpha \\
\beta \\
\hline
\alpha \& \beta \\
\end{array}
\]

Truth table:

<table>
<thead>
<tr>
<th>\alpha</th>
<th>\beta</th>
<th>\alpha &amp; \beta</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
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<td>F</td>
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<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>
Why Prove Soundness?

• Why use truth-table?
  Isn’t “soundness” obvious?

• Do you believe...

  Given
  Some $X$s are $Y$s &
  Some $Y$s are $Z$s
  conclude
  Some $X$s are $Z$s.

  Eg,

  Given
  Some polygons are rectangles &
  Some rectangles are squares
  conclude
  Some polygons are squares.

• If so...

  Given
  Some men are doctors &
  Some doctors are women
  conclude
  Some men are women?
# Sound Inference Rules

**[MP]** \( P \Rightarrow Q \)

\[
\begin{array}{c}
\frac{P}{Q}
\end{array}
\]

**[\forall E]** \( \forall x. \varphi(x) \)

\[
\frac{\varphi(A) \quad \text{for any } A}{\varphi(A)}
\]

**[\land I]** \( P \)

\[
\begin{array}{c}
P \\
Q
\end{array}
\]

\[
\frac{P \land Q}{P \land Q}
\]

**[\land E]** \( P \land Q \)

\[
\frac{P \land Q}{P}
\]

**[RC]** \( P \Rightarrow Q \)

\[
\begin{array}{c}
P \\
Q \Rightarrow R
\end{array}
\]

\[
\frac{P \Rightarrow R}{P \Rightarrow R}
\]

**[MG]** \( P \Rightarrow Q \)

\[
\begin{array}{c}
P \\
\neg P \Rightarrow Q
\end{array}
\]

\[
\frac{Q}{\text{Contradiction}}
\]

**[\lor D]** \( P \lor Q \)

\[
\begin{array}{c}
\neg P
\end{array}
\]

\[
\frac{Q}{Q}
\]

**[\land E]** \( P \land Q \)

\[
\frac{P}{P}
\]

**[\lor I]** \( P \)

\[
\frac{P}{P \lor Q}
\]

**[\land I]** \( P \land Q \)

\[
\frac{P \land Q}{P \lor Q}
\]

\[\ldots\]
Sound Rules of Inference

\[ KB_0 \xrightarrow{ir_j} KB_1 \]

Inference rule \( ir_j \) maps \( KB_0 \) into \( KB_1 \)

If \( ir_j \) is sound [aka Truth-Preserving], then

\[ KB_0 \models KB_1 \]

Ie, \( \mathcal{M}(KB_0) \subseteq \mathcal{M}(KB_1) \)

Hence: If believe \( KB_0 \), must believe \( KB_1 \).

Possible Worlds

\( \mathcal{M}(KB_0) \)

\( \mathcal{M}(KB_1) \)
Sound Rules of Inference – con’t

- In general, $KB_1 = KB_0 + \Delta$, so $KB_1 \models KB_0$

So $KB_1, KB_0$ hold in EXACTLY same worlds:

![Diagram showing possible worlds with $M(KB_0)$ and $M(KB_1)$]

- If each $ri_j$ is sound, then sequence $\langle ri_1 \cdots ri_n \rangle$ is sound.
Answering Queries

• Adding Truths (Forward Chaining)
  Given $KB_0$
  Produce $KB_N$ s.t.

  \[
  \begin{array}{c}
  KB_0 \quad \overset{r_1}{\leadsto} \quad KB_1 \quad \cdots \quad \overset{r_N}{\leadsto} \quad KB_N
  \end{array}
  \]

  (If \{ri\}_j sound, then $KB_0 \models KB_N$)

• Answering Questions (Backward Chaining)
  Given $KB_0$, $\sigma$
  Determine if $KB \models \sigma$

  Requires sound $\langle ri_j \rangle_j$ s.t.

  \[
  \begin{array}{c}
  KB_0 \quad \overset{r_1}{\leadsto} \quad KB_1 \quad \cdots \quad \overset{r_N}{\leadsto} \quad KB_N
  \end{array}
  \]

  and $\sigma \in KB_N$
Properties of Derivation Process

• $\vdash_\alpha$ is **SOUND**

$$\Sigma \vdash_\alpha \Psi \Rightarrow \Sigma \models \Psi$$

Produces only “true” results

• $\vdash_\alpha$ is **COMPLETE**

$$\Sigma \vdash_\alpha \Psi \iff \Sigma \models \Psi$$

Produces all “true” results

• $\vdash_\alpha$ is **DECIDABLE**

$$\Sigma \vdash_\alpha ? \Psi \text{ returns } Y \text{ or } N \text{ in finite time}$$
Degenerate $\vdash_{\alpha}$

- For any $\Sigma$, $\Psi \subseteq$ WFFs:
  
  $\Sigma \not\vdash_{N} \Psi$
  
  $\Sigma \vdash_{P} \Psi$

- Notice:
  
  - $\vdash_{N}$ is SOUND
    (everything it returns is logically entailed)
  
  - $\vdash_{P}$ is COMPLETE
    (it returns everything logically entailed)

  - $\vdash_{N}$, $\vdash_{P}$ are DECIDABLE
    (answer every question)
**Fundamental Limitation**

- For any sufficiently complicated domain as complex as arithmetic

- **NO** $\vdash_\alpha$ can be
  SOUND, COMPLETE, DECIDABLE!!

- Reduction to Halting Problem.

\[
\text{Not Predicate Calculus’s fault:} \\
\text{Reasoning is inherently undecidable,} \\
\text{no manner what formalism used.}
\]
Responses

• Deals only with WORST-Case!
  “Typical” case can be better.

TradeOffs (to increase efficiency):

? Sacrifice SOUNDness?
No — too severe.

? Sacrifice COMPLETEness?
Reasonable... Specific proposals:
  – Use only (incomplete set of) Inference Rules
  – Use complete set of Inference Rules,
    but limit depth (... stop expanding nodes. . .)

? Sacrifice EXPRESSIVENess?

[EXPRESSIVENess ≈ what can be distinguished.]
Common approach!
(After all, Logic’s distinctions caused problems!)
Disallow “∨” “¬” “∃” ...
Implemented Systems

- DataBase Systems
  \(\approx\) Sound, Complete, Limited Expressiveness

- Prolog
  \(\approx\) Sound, Complete, Limited Expressiveness

- General Theorem Provers
  Sound, Complete, Complete Expressiveness

- Production System (Emycin, OPS)
  \(\approx\) Sound, \(\approx\) Complete, Limited Expressiveness

- Frame Systems
  \(?\) Sound?, \(?\) Complete?, Limited Expressive

- Description Languages
  Sound, Complete, Limited Expressiveness

- Truth Maintenance
• Complexity Cliffs:

Be as expressive as possible within “tractable” side

Frames/SemanticNets tractable, but not very expressive

full Predicate Calculus is very expressive by not tractable
Description Logics

- Undecidable if $\forall, \exists, \neg, \lor, \land$

- Just use “tractable” subset
  eg, avoid $\neg$, only some types of $\lor, \ldots$

- Define concepts
  \begin{align*}
  \text{HappyMother} & \equiv \\
  & \text{Woman whose children are all RICH, and all married to pediatricians.} \\
  \text{SuccessfulParent} & \equiv \text{PERSON whose DAUGHTERS are married to doctors.} \\
  \\
  \text{“Subsumption” questions like:} & \\
  \text{Is every HappyMother a SuccessfulParent?} \\
  \text{What if...} & \\
  \text{at least 3 daughters?} & \\
  \text{either MDs or Profs?} \\
  \end{align*}

Uses: CLASSIC (AT&T)
  Financial Management, Database Interfaces, Software Information Systems
How to Reason?

Q: How to reason?
   Given $KB$, $q$, determine if $KB \models q$?

A: Select Inference Rule $IR$
   Select Fact(s) $\{F_i\}$ from $KB$
   Apply rule $IR$ to Facts $\{F_i\}$
      to get new Fact $\phi$
      ...Add $\phi$ to $KB$
   Repeat until find $\phi = q$

Issues:
1. Lots of Inference Rules
   Which one to use, when?
2. Is overall system “complete”? 
   If $\exists$ answer, guaranteed to find it?
Notation

• $\models_m \phi$
  \textit{m} is model of formula $\phi$
  Model (interpretation, possible world) $m$
  assigns formula $\phi$ to $+$

• Formula is \textbf{satisfiable} if it has $\geq 1$ model

• Formula is \textbf{tautology}
  ("valid" in predicate logic)
  if true in every interpretation
  \begin{align*}
  & a \lor \neg a; \quad a \Rightarrow a \\
  \end{align*}

• A formula is a \textbf{contradiction}
  (aka "inconsistent")
  if false in every interpretation,
  \begin{align*}
  & a \land \neg a; \quad \{a, \ a \Rightarrow b, \ \neg b\} \\
  \end{align*}

• $\Sigma \models \phi$ means
  Whenever $\Sigma$ is interpreted true, so is $\phi$
  \begin{align*}
  & \ldots \quad \phi \text{ is true in every model of } \Sigma.
  \end{align*}