The three column Bandpass problem is solvable in linear time

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Abstract

The general Bandpass problem is NP-hard and was claimed NP-hard when the number of columns is three. Previously we designed a polynomial time row-stacking algorithm for the three column case, to produce a solution that is at most 1 less than the optimum. We show in this paper that for any bandpass number $B \geq 2$, an optimal solution is always achievable in linear time.

Keywords: Bandpass problem, exact algorithm, polynomial time solvable

1 Introduction

The *Bandpass problem* can be described as follows [2, 1, 3]: Given a binary matrix A of dimension $m \times n$, and a positive integer B called the *bandpass number*, a set of B consecutive non-zero elements in a column of the matrix is called a *bandpass*. When counting the number of bandpasses, no two of them in the same column are allowed to have common rows. The goal of the problem is to find an optimal permutation of rows of the matrix such that the total number of extracted bandpasses is maximized.

This combinatorial optimization problem arises in optical communication networks, where the goal is to design an optimal packing of information flows on different wavelengths into groups such that the highest available cost reduction can be obtained using wavelength division multiplexing technology [1]. In such an application, the input binary matrix $A_{m \times n}$ represents a sending point which has m information packages to be sent to n different destination points, where $a_{ij} = 1$ if information package i is *not* destined for point j, or $a_{ij} = 0$ otherwise. Essentially, B consecutive 1's indicate an opportunity for merging information and thus reducing the communication cost. Though multiple bandpass numbers can be used in practice, for the sake of complexities and costs, usually only one fixed bandpass number is considered [1].

The general Bandpass problem, for any fixed $B \geq 2$, has been proven to be NP-hard [1, 3], and can be approximated to some extent [3]. The Bandpass problem was firstly incorrectly proven to be NP-hard in [1], for all $n \geq 3$, where a reduction to the Integer Programming problem then to the 3SAT problem was used. (In fact, the restricted decision version of the Bandpass problem "proven to be NP-complete", in which there are $B+2$ rows, can be easily solved by noticing that the *yes*-instances should contain at least $B-2$ rows of all-1's.) A correct NP-hardness proof involves a reduction from the well-known *Hamiltonian path* problem, where the constructed matrix A has more columns (corresponding to edges) than rows (corresponding to vertices). Thus, it would be interesting to investigate the special yet practical capacitated broadcasting case in which the number of columns/destinations, n, is bounded. To this purpose, Lin proposed a row-stacking algorithm, which produces an optimal solution when $n = 1, 2$, and produces a solution that is at most 1 less than the optimum when $n = 3$ [3]. In this paper, we show that the three column Bandpass problem is solvable in linear time for any $B \geq 2$, by a modified row-stacking algorithm to take care of the exceptional cases that cannot be handled by the original row-stacking algorithm.

Outline of the presentation. In the next section, we summarize the row-stacking algorithm originally presented in [3], and the cases where it does and where it does not guarantee an optimal solution, respectively. Among the latter category of cases where the row-stacking algorithm does not guarantee an optimal solution, we prove for some of them that the row-stacking solution is actually optimal, and for the other we modify the algorithm a little bit to produce an optimal solution.

The rest of paper presentation involves case by case analysis on the input instance. We have tried hard to group them and wish to deliver a short argument covering all distinct cases. However such a short argument is seemingly unlikely, we are able to pinpoint two extreme cases, presented in Section 3, to which several other cases can be reduced to prove the optimality of the row-stacking solution. Following the notations introduced in the next section, Sections 4 and 5 discuss the cases when $r_6 = 0$ and when $r_6 \neq 0$, respectively. Within Section 4, Subsections 4.1 and 4.2 consider the subcases where $q_6 = 0$ and where $q_6 \neq 0$, respectively; In Section 5, Subsection 5.1 is for the existence of a zero among r_2, r_4, r_8; Subsection 5.2 is for the non-existence of a zero among r_2, r_4, r_8.

2 The preliminaries

Let n denote the number of columns in the Bandpass problem, and m the number of rows. The main result in this paper is an $O(m)$-time exact algorithm for $n = 3$, thus disproving the claim that *the Bandpass problem with* $n \geq 3$ *is NP-hard* made in [1]. Previously, exact algorithms were proposed for $n = 1$ and $n = 2$ [1], which were formalized into the row-stacking algorithm [3]. This row-stacking algorithm can also be applied for $n = 3$ to produce a row permutation achieving the maximum number of bandpasses, or 1 less.

In more details, for $n = 1$, the row-stacking algorithm puts all non-zero rows consecutively, which is an optimal permutation, no matter what B is; For $n = 2$, the row-stacking algorithm firstly classifies rows into (0 0)-, (0 1)-, (1 0)-, and (1 1)-rows, then stacks them in the order of (1 0)-rows, then (1 1)-rows, then (0 1)-rows, lastly (0 0)-rows; this gives an optimal row permutation, again no matter what B is, since all 1's in each column are placed consecutively.

For $n = 3$, there are eight types of rows: (0 0 0)-, (0 0 1)-, (0 1 0)-, (0 1 1)-, (1 0 0)-, (1 0 1)-, (1 1 0)-, and (1 1 1)-rows; and assume there are $m_1, m_2, \ldots, m_8$ of them respectively. The row-stacking algorithm stacks them in the order of firstly (1 0 0)-rows, then sequentially (1 0 1)-rows, (1 1 1)-rows, (1 1 0)-rows, (0 1 0)-rows, (0 1 1)-rows, (0 0 1)-rows, and lastly (0 0 0)-rows (see

Figure 1). In this row placement, the 1's in each of the first two columns appear consecutively,

row type in order	quantity
$(0,0,0)$	m_1
$(0,0,1)$	m_2
$(0,1,1)$	m_3
$(0,1,0)$	m_4
$(1,1,0)$	m_5
$(1,1,1)$	m_6
$(1,0,1)$	m_7
$(1,0,0)$	m_8

Figure 1: The row placement produced by the row-stacking algorithm.

but the 1's in the third column could be separated into two bands. Therefore, the number of bandpasses in this row-stacking solution differs the optimum by at most 1. Since $(0,0,0)$-rows do not contribute to bandpasses, we ignore them hereafter. Further let $m_i = q_i B + r_i$, where q_i, r_i are the quotient and remainder of dividing B into m_i, for $i = 2, \ldots, 8$, and

$$MAX = \left\lfloor \frac{m_5 + m_6 + m_7 + m_8}{B} \right\rfloor + \left\lfloor \frac{m_3 + m_4 + m_5 + m_6}{B} \right\rfloor + \left\lfloor \frac{m_2 + m_3 + m_6 + m_7}{B} \right\rfloor, \tag{2.1}$$

which is an upper bound on the number of bandpasses that can ever be generated. Since the number of bandpasses in the row-stacking solution is

$$\left\lfloor \frac{m_5 + m_6 + m_7 + m_8}{B} \right\rfloor + \left\lfloor \frac{m_3 + m_4 + m_5 + m_6}{B} \right\rfloor + \left\lfloor \frac{m_2 + m_3}{B} \right\rfloor + \left\lfloor \frac{m_6 + m_7}{B} \right\rfloor \geq MAX - 1,$$

we have

$$MAX \geq OPT \geq MAX - 1, \tag{2.2}$$

where OPT denotes the number of bandpasses in the optimal solution. This proves the following lemma.

Lemma 1 [3] *The three column Bandpass problem can be solved almost exactly in linear time, to obtain a row permutation generating either the maximum number of, or one less, bandpasses.*

One can see that there are many cases where the row-stacking solution is optimal as it generates MAX bandpasses, for example, when $m_4 + m_5 = 0$, or when $m_2 + m_3$ is a multiple of B, or when $(m_2 + m_3)\%B + (m_6 + m_7)\%B < B$ (% is the modulo operation).

Additionally, since bandpasses are column independent, one may permute the three columns arbitrarily without affecting the extracted bandpasses. It follows that we have six distinct column permutations to run the row-stacking algorithm, and thus six solutions. Among these six solutions, if their resultant numbers of bandpasses consist of two distinct values, which must be MAX and $MAX - 1$, then the solution associated with the larger number of bandpasses must be optimal; It is unclear whether or not at least one of these six solutions is optimal for any $B \geq 2$ [3].

In fact, the unsure case is all six row-stacking solutions associated with the six column permutations generate $MAX - 1$ bandpasses. For column permutation $(1, 2, 3)$, since the sizes of the two

1-bands in the third column are $m_2 + m_3$ and $m_6 + m_7$, respectively (see Figure 1), this means that $(r_2 + r_3)\%B > 0$, $(r_6 + r_7)\%B > 0$, and $(r_2 + r_3)\%B + (r_6 + r_7)\%B \geq B$. (Note that there must be $m_4 + m_5 > 0$ too, which however is implied from other column permutations.) Likewise, we can derive from the other five column permutations similar constraints on the r_i's, which are summarized in Table 1.

Column permutation	Sizes of two 1-bands in the third column	Sizes modulo B
$(1, 2, 3)$	$m_2 + m_3$, $m_6 + m_7$	$(r_2 + r_3)\%B$, $(r_6 + r_7)\%B$
$(2, 1, 3)$	$m_2 + m_7$, $m_6 + m_3$	$(r_2 + r_7)\%B$, $(r_6 + r_3)\%B$
$(1, 3, 2)$	$m_4 + m_3$, $m_6 + m_5$	$(r_4 + r_3)\%B$, $(r_6 + r_5)\%B$
$(3, 1, 2)$	$m_4 + m_5$, $m_6 + m_3$	$(r_4 + r_5)\%B$, $(r_6 + r_3)\%B$
$(2, 3, 1)$	$m_8 + m_7$, $m_6 + m_5$	$(r_8 + r_7)\%B$, $(r_6 + r_5)\%B$
$(3, 2, 1)$	$m_8 + m_5$, $m_6 + m_7$	$(r_8 + r_5)\%B$, $(r_6 + r_7)\%B$

Table 1: The sizes, and modulo B, of two 1-bands in the third column of the row-stacking solutions for the six column permutations.

In the next three sections, we investigate the above unsure case to identify the subcases where MAX bandpasses can be achieved, and prove for the other subcases that $OPT = MAX - 1$, meaning that the row-stacking solutions are already optimal. The exact algorithm essentially returns an optimal row permutation with MAX bandpasses when the input instance falls into the identified subcases, or otherwise returns either of the six row-stacking solutions.

3 Two extreme subcases

Lemma 2 *When $m_2, m_4, m_6, m_8 = 0$, $r_3 + r_5 \geq B$, $r_5 + r_7 \geq B$, $r_7 + r_3 \geq B$, and $r_3 + r_5 + r_7 < 2B$, then $OPT = MAX - 1$.*

PROOF. Recall that $m_i = q_i B + r_i$, for $i = 3, 5, 7$. We first show that if one of q_3, q_5, q_7 is zero, then $OPT = MAX - 1$. Without loss of generality, assume $q_7 = 0$. From the lemma premises, we have $MAX = 2q_3 + 2q_5 + 3$, and if there were an optimal row placement $\mathcal{P}$ achieving MAX bandpasses, then there are $q_5 + 1$, $q_3 + q_5 + 1$, $q_3 + 1$ bandpasses in the first, second, third columns of $\mathcal{P}$, respectively.

Since the total number of rows is $m_3 + m_5 + m_7 < (q_3 + q_5 + 2)B$, we conclude that in $\mathcal{P}$ there must be some bandpasses in the first column overlap (that is, share rows) with bandpasses in the third column; but none in the first column would overlap with two bandpasses in the third column due to the non-existence of $(1, 1, 1)$-rows. Note that since $m_7 = r_7 < B$, each overlapping region contains at most r_7 rows. Equivalently, there are pairs of overlapping bandpasses, one in the first column and one in the third column, and these overlapping regions, consisting of solely $(1, 0, 1)$-rows, separate the rows of $\mathcal{P}$ into chunks. For every bandpass (in the first or the third column) participating in the overlapping pairs, if a port of it belongs to a chunk, then the bandpass is said to belong to the chunk. Because there are $q_3 + q_5 + 1$ bandpasses in the second column of $\mathcal{P}$, we conclude that there is (at least) one chunk in which the number of bandpasses in the second column is strictly less than the total number of bandpasses in the first and the third columns.

Recall that inside a chunk, no bandpass in the first column would overlap with any bandpass in the third column. It follows that in this chunk strictly greater than $B - r_7$ 1's in the second column are not involved in any bandpasses. Nevertheless, in order to achieve MAX bandpasses, at most $(r_3 + r_5) - B$ 1's in the second column of $\mathcal{P}$ can sit outside of generated bandpasses. This is a contradiction since $(r_3 + r_5) - B < B - r_7$. Such a contradiction, together with Eq. (2.2), implies that $OPT = MAX - 1$.

When all q_3, q_5, q_7 are positive, and assume to the contrary that $OPT = MAX = 2q_3 + 2q_5 + 2q_7 + 3$ is achieved in a row placement $\mathcal{P}$, then we examine where the topmost bandpass is in $\mathcal{P}$. Assume without loss of generality that it occurs in the first column, then the second topmost bandpass should not occurs in the first column, for otherwise at least B 1's would not be involved in any generated bandpasses in $\mathcal{P}$. Again assume without loss of generality that the second topmost bandpass occurs in the second column. These two bandpasses must overlap for the same reason above. Due to the non-existence of $(1,1,1)$-rows, the third topmost bandpass does not overlap with the topmost bandpass. Assume there are ℓ $(1,0,1)$-rows in the topmost bandpass. If we take away the B rows in the topmost bandpass from the instance, the resultant new instance I' contains $m'_3 = m_3$ $(0,1,1)$-rows, $m'_5 = m_5 - B + \ell$ $(1,1,0)$-rows, and $m'_7 = m_7 - \ell$ $(1,0,1)$-rows. Apparently $\ell \leq (r_3 + r_7)\%B = r_3 + r_7 - B$, implying that $r'_7 = r_7 - \ell \geq B - r_3 > 0$, $r'_5 = r_5 + \ell \leq r_3 + r_5 + r_7 - B < B$, $r'_3 + r'_5 = r_3 + r_5 + \ell \geq B$, $r'_5 + r'_7 = r_5 + r_7$, $r'_7 + r'_3 \geq B$, and $r'_3 + r'_5 + r'_7 = r_3 + r_5 + r_7$. This new instance I' satisfies the premises in the lemma, with B less rows than the original instance and again with $OPT(I') = MAX(I')$.

It follows that if we were to apply the same reduction procedure, we will eventually end up with an instance which satisfies the premises in the lemma and with $OPT = MAX$, but one of q_3, q_5, q_7 is zero. This is a contradiction to the fact proven in the first half. Therefore, for all instances satisfying the premises, their optimal row placement contains only $MAX - 1$ bandpasses, suggesting that the row-stacking solutions are already optimal. This proves the lemma. $\square$

Lemma 3 *When* $m_3, m_5, m_7 = 0$, $r_2 + r_6 \geq B$, $r_4 + r_6 \geq B$, $r_8 + r_6 \geq B$, $r_2 + r_4 + r_6 < 2B$, $r_4 + r_8 + r_6 < 2B$, $r_8 + r_2 + r_6 < 2B$, *if* $r_2 + r_4 + r_8 + 2r_6 < 3B$ *or* $q_2, q_4, q_8 = 0$, *then* $OPT = MAX - 1$.

Proof. Recall that $m_i = q_i B + r_i$, for $i = 2, 4, 8, 6$. From the lemma premises and Eq. (2.1), we have $MAX = q_2 + q_4 + q_8 + 3q_6 + 3$, and if there were an optimal row placement $\mathcal{P}$ achieving MAX bandpasses, then there are $q_8 + q_6 + 1$, $q_4 + q_6 + 1$, $q_2 + q_6 + 1$ bandpasses in the first, second, third columns of $\mathcal{P}$, respectively.

Since $(0,0,1)$-rows are not involved in any bandpasses formed in the first and the second columns, these bandpasses must overlap at least $(q_8 + q_6 + 1 + q_4 + q_6 + 1)B - (m_4 + m_6 + m_8) = q_6 B + 2B - r_4 - r_6 - r_8$ rows. These rows have 1 in both the first and the second column, and thus must be $(1,1,1)$-rows. If one of these $(1,1,1)$-rows is involved in a bandpass generated in the third column, that is, there are three bandpasses, one from each column, overlapping at a $(1,1,1)$-row, then there are B consecutive $(1,1,1)$-rows in the optimal placement (which includes the shared $(1,1,1)$-row). Removing these B consecutive $(1,1,1)$-rows, on one hand we obtain a reduced instance I' for which all the premises hold except that q_6 decreases by 1; on the other hand, we obtain a row placement for I' achieving $MAX(I') = MAX - 3$ bandpasses. It follows that by repeatedly reducing the instances whenever possible, we may assume without loss of generality that none of the $q_6B + 2B - r_4 - r_6 - r_8$ $(1,1,1)$-rows is involved in any bandpasses in the third column. Consequently, the maximum possible number of bandpasses in the third column becomes

$$\left\lfloor \frac{m_2 + m_6 - (q_6B + 2B - r_4 - r_6 - r_8)}{B} \right\rfloor = q_2 + \left\lfloor \frac{r_2 + r_4 + r_8 + 2r_6 - 2B}{B} \right\rfloor.$$

Therefore, if $r_2 + r_4 + r_8 + 2r_6 < 3B$, this maximum possible number is $q_2 < q_2 + q_6 + 1$, a contradiction.

Note that $r_2 + r_4 + r_8 + 2r_6 < 4B$. Therefore, if $q_2, q_4, q_8 = 0$, this maximum possible number is $1 \leq q_6 + 1$ and the equality holds only when $q_6 = 0$. In such a case, the bandpass in the third column may overlap with at most one of the bandpass in the first column and the bandpass in the second column, a contradiction to the fact that these three bandpasses must pairwise overlap. Hence, for all instances satisfying the premises, their optimal row placement contains only $MAX - 1$ bandpasses. This proves the lemma. $\square$

4 When $r_6 = 0$

We separate into two disjoint cases according to whether or not $q_6 = 0$. One can verify that since $r_6 = 0$, Table 1 reduces to the following Table 2. Furthermore, $m_2 + r_3 > B$ if and only if $m_2 + r_7 > B$, $m_4 + r_3 > B$ if and only if $m_4 + r_5 > B$, and $m_8 + r_5 > B$ if and only if $m_8 + r_7 > B$.

Column permutations	Sizes of two 1-bands modulo B
$(1,2,3)$	$(r_2 + r_3)\%B$, r_7
$(2,1,3)$	$(r_2 + r_7)\%B$, r_3
$(1,3,2)$	$(r_4 + r_3)\%B$, r_5
$(3,1,2)$	$(r_4 + r_5)\%B$, r_3
$(2,3,1)$	$(r_8 + r_7)\%B$, r_5
$(3,2,1)$	$(r_8 + r_5)\%B$, r_7

Table 2: The sizes modulo B of the two 1-bands in the third column in the row-stacking solutions when $r_6 = 0$.

4.1 When $q_6 = 0$

We consider a few subcases. In the first subcase (Case 1.1), $m_2 + r_3 > B$. It follows that $m_2 > B - r_3 > 0$. We stack in order $m_2 - (B - r_3)$ $(0,0,1)$-rows, then all $(1,0,1)$-rows, all $(1,0,0)$-rows, all $(1,1,0)$-rows, all $(0,1,0)$-rows, all $(0,1,1)$-rows, and lastly the other $B - r_3$ $(0,0,1)$-rows. In the resultant row permutation (see Figure 2(a)), all 1's in each of the first two columns are consecutive, and the first one of the two 1-bands in the third column has size $(q_3 + 1)B$. It is therefore an optimal solution. Symmetrically, if $m_4 + r_3 > B$ or $m_8 + r_5 > B$, we are also able to obtain an optimal row permutation achieving MAX bandpasses. Therefore, in the sequel we assume that

row type	quantity
$(0,0,1)$	$B - r_3$
$(0,1,1)$	m_3
$(0,1,0)$	m_4
$(1,1,0)$	m_5
$(1,0,0)$	m_8
$(1,0,1)$	m_7
$(0,0,1)$	$m_2 - (B - r_3)$

(a) Case 1.1

row type	quantity
$(1,0,1)$	$m_7 - (B - r_5 - r_8)$
$(0,0,1)$	r_2
$(0,1,1)$	m_3
$(0,1,0)$	r_4
$(1,1,0)$	m_5
$(1,0,0)$	r_8
$(1,0,1)$	$B - r_5 - r_8$

(b) Case 1.2

row type	quantity
$(0,0,1)$	r_2
$(0,1,1)$	$B - r_2$
$(1,1,0)$	$m_5 - (B - r_2 - r_3 - r_4)$
$(1,0,0)$	r_8
$(1,0,1)$	m_7
$(0,1,1)$	$m_3 - (B - r_2)$
$(0,1,0)$	r_4
$(1,1,0)$	$B - r_2 - r_3 - r_4$

(c) Case 1.4

row type	quantity
$(0,0,1)$	r_2
$(0,1,1)$	$B - r_2$
$(0,1,0)$	$r_4 - (B - r_2 - r_3)$
$(1,1,0)$	m_5
$(1,0,0)$	r_8
$(1,0,1)$	m_7
$(0,1,1)$	$m_3 - (B - r_2)$
$(0,1,0)$	$B - r_2 - r_3$

(d) Case 1.4

Figure 2: The optimal row placements when $m_6 = 0$.

$q_2, q_4, q_8 = 0$, replacing m_2, m_4, m_8 by r_2, r_4, r_8 respectively, and that $r_2 + r_3 < B$, $r_4 + r_3 < B$, and $r_8 + r_5 < B$ (implying $r_2 + r_7 < B$, $r_4 + r_5 < B$, and $r_8 + r_7 < B$).

From $r_5 + r_7 + r_8 \geq B$ we conclude that $r_7 \geq B - r_5 - r_8 > 0$. If $r_2 + r_3 + r_5 + r_7 + r_8 \geq 2B$ (Case 1.2), we stack in order $B - r_5 - r_8$ $(1,0,1)$-rows, then all $(1,0,0)$-rows, all $(1,1,0)$-rows, all $(0,1,0)$-rows, all $(0,1,1)$-rows, all $(0,0,1)$-rows, and lastly the other $m_7 - (B - r_5 - r_8)$ $(1,0,1)$-rows. In the resultant row permutation (see Figure 2(b)), the second one of the two 1-bands in the first column has size $(q_5 + 1)B$, all 1's in the second column are consecutive, and in the third column, the size of the first one of the two 1-bands is $(q_3 + q_7 - 1)B + (r_2 + r_3 + r_5 + r_7 + r_8) \geq (q_3 + q_7 + 1)B$, thus achieving the maximum possible number of bandpasses. We conclude this row permutation is optimal. Symmetrically, if $r_2 + r_3 + r_5 + r_7 + r_4 \geq 2B$ or $r_4 + r_3 + r_5 + r_7 + r_8 \geq 2B$, we are also able to obtain an optimal row permutation achieving MAX bandpasses.

Next, if $r_2 + r_4 + r_8 + r_3 + r_5 + r_7 < 2B$ (Case 1.3), we convert all $(0,0,1)$-rows to $(0,1,1)$-rows, all $(0,1,0)$-rows to $(1,1,0)$-rows, and all $(1,0,0)$-rows to $(1,0,1)$-rows, by adding 1's. This reduces the original instance I into a new instance I' such that $OPT(I) \leq OPT(I')$. By Lemma 2, we have $OPT(I') = MAX(I') - 1 = 2(q_3 + q_5 + q_7) + 2 = MAX(I) - 1 \leq OPT(I)$, we conclude that $OPT(I) = MAX(I) - 1$, indicating that all the six row-stacking solutions are already optimal.

If $r_2 + r_4 + r_8 + r_3 + r_5 + r_7 \geq 2B$ and $q_3 > 0$ (Case 1.4), we check whether or not $r_2 + r_3 + r_4 < B$. If so, we stack in order $B - r_2 - r_3 - r_4$ $(1,1,0)$-rows, then all $(0,1,0)$-rows, $m_3 - (B - r_2)$ $(0,1,1)$-rows, all $(1,0,1)$-rows, all $(1,0,0)$-rows, the other $m_5 - (B - r_2 - r_3 - r_4)$ $(1,1,0)$-rows, the other $B - r_2$

$(0,1,1)$-rows, and lastly all $(0,0,1)$-rows. The resultant row permutation is shown in Figure 2(c). If $r_2 + r_3 + r_4 \geq B$, that is, $r_4 \geq B - r_2 - r_3$, we stack in order $B - r_2 - r_3$ $(0,1,0)$-rows, then $m_3 - (B - r_2)$ $(0,1,1)$-rows, all $(1,0,1)$-rows, all $(1,0,0)$-rows, all $(1,1,0)$-rows, the other $m_4 - (B - r_2 - r_3)$ $(0,1,0)$-rows, the other $B - r_2$ $(0,1,1)$-rows, and lastly all $(0,0,1)$-rows. The resultant row permutation is shown in Figure 2(d). In both row permutations, the size of the second one of the two 1-bands in the second column is a multiple of B, and the size of the first one of the two 1-bands in the third column is B. Since all 1's in the first column of Figure 2(d) are consecutive, it is optimal; For Figure 2(c), the size of the first one of the two 1-bands in the first column is $(q_5+q_7-1)B+r_2+r_4+r_8+r_3+r_5+r_7 \geq (q_5+q_7+1)B$, thus achieving the maximum possible number of bandpasses. Symmetrically, if $q_5 > 0$ or $q_7 > 0$, we are also able to obtain an optimal row permutation achieving MAX bandpasses.

If $q_3, q_5, q_7 = 0$ (Case 1.5, and replacing m_3, m_5, m_7 by r_3, r_5, r_7 respectively), then $MAX = 1+1+1 = 3$. If there were one bandpass in each of the three columns, from $r_2+r_3+r_5+r_7+r_8 < 2B$, $r_2+r_3+r_5+r_7+r_4 < 2B$, and $r_4+r_3+r_5+r_7+r_8 < 2B$ we conclude that these three bandpasses must pairwise overlap. However, such an overlapping scenario would imply the existence of $(1,1,1)$-rows, a contradiction. Therefore, in this case, $OPT = MAX - 1$, and all the six row-stacking solutions are already optimal.

4.2 When $q_6 > 0$

Since the number of $(1,1,1)$-rows is a multiple of B, we only need to consider the scenarios (Case 1.3 and Case 1.5) in the last section for $q_6 = 0$ where $OPT = MAX-1$. In particular, in these scenarios we have $q_2, q_4, q_8 = 0$ (and replacing m_2, m_4, m_8 by r_2, r_4, r_8 respectively), $r_2+r_4+r_3+r_5+r_7 < 2B$, $r_2+r_8+r_3+r_5+r_7 < 2B$, and $r_4+r_8+r_3+r_5+r_7 < 2B$ (implying $r_2+r_3 < B$, $r_2+r_7 < B$, $r_4+r_3 < B$, $r_4+r_5 < B$, $r_8+r_5 < B$, and $r_2+r_7 < B$).

When $r_4 = 0$ (Case 2.1), we stack in order all $(1,0,0)$-rows, then all $(1,0,1)$-rows, $m_6 - r_7$ $(1,1,1)$-rows, all $(1,1,0)$-rows, the other r_7 $(1,1,1)$-rows, all $(0,1,1)$-rows, and lastly all $(0,0,1)$-rows. In the resultant row permutation (see Figure 3(a)), all 1's in the first two columns are consecutive respectively, and the size of second one of the two 1-bands in the third column is $(q_6+q_7)B$. It is therefore an optimal solution. Symmetrically, when $r_2 = 0$ or $r_8 = 0$, we are also able to obtain an optimal row permutation achieving MAX bandpasses.

When $m_3 \geq r_8$ (Case 2.2), we stack in order all $(0,0,1)$-rows, then $m_3 - r_8$ $(0,1,1)$-rows, $r_7 + r_8$ $(1,1,1)$-rows, all $(1,1,0)$-rows, all $(0,1,0)$-rows, r_8 $(0,1,1)$-rows, $m_6 - (r_7+r_8)$ $(1,1,1)$-rows, all $(1,0,1)$-rows, and lastly all $(1,0,0)$-rows. In the resultant row permutation (see Figure 3(b)), there are two 1-bands in the first column, of which the first has size q_6B; the second column has only one 1-band; and the first one of the two 1-bands in the third column has size $(q_6+q_7)B$. It is therefore an optimal solution. Symmetrically, if $m_5 \geq r_2$ or $m_7 \geq r_4$, we are able to achieve an optimal row permutation.

In the remaining case, we have $q_3, q_5, q_7 = 0$ (and replacing m_3, m_5, m_7 by r_3, r_5, r_7 respectively), $r_3 < r_8$, $r_5 < r_2$, and $r_7 < r_4$. If $r_3+r_5+r_7 \geq B$ (Case 2.3, that is, $r_5 \geq B-r_3-r_7 \geq B-r_8-r_7 > 0$), we stack in order all $(1,0,1)$-rows, the $m_6 - r_7$ $(1,1,1)$-rows, all $(0,1,0)$-rows, $m_5 - (B - r_3 - r_7)$ $(1,1,0)$-rows, all $(1,0,0)$-rows, the other $B - r_3 - r_7$ $(1,1,0)$-rows, the other r_7 $(1,1,1)$-rows, all $(0,1,1)$-rows, and lastly all $(0,0,1)$-rows. In the resultant row permutation (see Figure 3(c)), there are two 1-bands in each column, of which (the second, the first, and the second, respectively) one

row type	quantity
$(0,0,1)$	r_2
$(0,1,1)$	m_3
$(1,1,1)$	r_7
$(1,1,0)$	m_5
$(1,1,1)$	$m_6 - r_7$
$(1,0,1)$	m_7
$(1,0,0)$	r_8

(a) Case 2.1

row type	quantity
$(1,0,0)$	r_8
$(1,0,1)$	m_7
$(1,1,1)$	$m_6 - (r_7 + r_8)$
$(0,1,1)$	r_8
$(0,1,0)$	r_4
$(1,1,0)$	m_5
$(1,1,1)$	$r_7 + r_8$
$(0,1,1)$	$m_3 - r_8$
$(0,0,1)$	r_2

(b) Case 2.2

row type	quantity
$(0,0,1)$	r_2
$(0,1,1)$	r_3
$(1,1,1)$	r_7
$(1,1,0)$	$B - r_3 - r_7$
$(1,0,0)$	r_8
$(1,1,0)$	$r_5 - (B - r_3 - r_7)$
$(0,1,0)$	r_4
$(1,1,1)$	$m_6 - r_7$
$(1,0,1)$	r_7

(c) Case 2.3

Figure 3: The optimal row placements when $m_6 = q_6 B > 0$.

has size of a multiple of B. It is therefore an optimal solution.

If $r_3 + r_5 + r_7 < B$ (Case 2.4), we convert all $(0,1,1)$-rows, all $(1,1,0)$-rows, and all $(1,0,1)$-rows into $(1,1,1)$-rows by adding 1's. This reduces the original instance I into a new instance I' such that $OPT(I) \leq OPT(I')$. By Lemma 3 (the second case), we have $OPT(I') = MAX(I') - 1 = 3q_6 + 2 = MAX(I) - 1 \leq OPT(I)$, we conclude that $OPT(I) = MAX(I) - 1$, indicating that all the six row-stacking solutions are already optimal.

5 When $r_6 > 0$

This case is a bit more complex than the case of $r_6 = 0$. We separate into two disjoint subcases according to whether there is a zero in $\{r_2, r_4, r_8\}$.

5.1 When $r_2 \cdot r_4 \cdot r_8 = 0$

In this case, there is at least one zero among r_2, r_4, r_8. We assume without loss of generality that $r_4 = 0$, and thus Table 1 reduces to the following Table 3.

Column permutations	Sizes of two 1-bands modulo B
$(1,2,3)$	$(r_2 + r_3)\%B$, $(r_6 + r_7)\%B$
$(2,1,3)$	$(r_2 + r_7)\%B$, $(r_6 + r_3)\%B$
$(1,3,2)$	r_3, $(r_6 + r_5)\%B$
$(3,1,2)$	r_5, $(r_6 + r_3)\%B$
$(2,3,1)$	$(r_8 + r_7)\%B$, $(r_6 + r_5)\%B$
$(3,2,1)$	$(r_8 + r_5)\%B$, $(r_6 + r_7)\%B$

Table 3: The sizes modulo B of the two 1-bands in the third column in the row-stacking solutions when $r_6 > 0$ and $r_4 = 0$.

5.1.1 When $q_4 = 0$

In this case, there are no $(0,1,0)$-rows to be considered. If $m_2 + r_7 > B$ (Case 3.1), then $m_2 > B - r_7 > 0$. We stack in order $B - r_7$ $(0,0,1)$-rows, then all $(1,0,1)$-rows, all $(1,0,0)$-rows, all $(1,1,0)$-rows, all $(1,1,1)$-rows, all $(0,1,1)$-rows, and lastly the other $m_2 - (B - r_7)$ $(0,0,1)$-rows. In the resultant row permutation (see Figure 4(a)), all 1's in each of the first two columns are consecutive, and the second one of the two 1-bands in the third column has size $(q_7 + 1)B$. It is therefore an optimal solution. Symmetrically, if $m_8 + r_7 > B$, we are also able to obtain an optimal row permutation achieving MAX bandpasses. Therefore, in the sequel we assume that $q_2, q_8 = 0$, replacing m_2, m_8 by r_2, r_8 respectively, and that $r_2 + r_7 < B$ and $r_8 + r_7 < B$.

row type	quantity
$(0,0,1)$	$m_2 - (B - r_7)$
$(0,1,1)$	m_3
$(1,1,1)$	m_6
$(1,1,0)$	m_5
$(1,0,0)$	m_8
$(1,0,1)$	m_7
$(0,0,1)$	$B - r_7$

(a) Case 3.1

row type	quantity
$(1,0,0)$	r_8
$(1,1,0)$	m_5
$(1,1,1)$	$m_6 - (B - r_3)$
$(1,0,1)$	m_7
$(1,1,1)$	$B - r_3$
$(0,1,1)$	m_3
$(0,0,1)$	r_2

(b) Case 3.2

row type	quantity
$(0,0,1)$	$B - r_3 - r_6$
$(0,1,1)$	m_3
$(1,1,1)$	r_6
$(1,1,0)$	m_5
$(1,0,0)$	r_8
$(1,0,1)$	m_7
$(0,0,1)$	$r_2 - (B - r_3 - r_6)$

(c) Case 3.3

row type	quantity
$(1,0,1)$	$B - r_2 - r_3 - r_6$
$(0,0,1)$	r_2
$(0,1,1)$	m_3
$(1,1,1)$	r_6
$(1,1,0)$	m_5
$(1,0,0)$	r_8
$(1,0,1)$	$m_7 - (B - r_2 - r_3 - r_6)$

(d) Case 3.4

Figure 4: The optimal row placements when $r_6 > 0$ and $m_4 = 0$.

If $m_6 + r_3 > B$ (Case 3.2), then $m_6 > B - r_3 > 0$. We stack in order all $(0,0,1)$-rows, then all $(0,1,1)$-rows, $B - r_3$ $(1,1,1)$-rows, all $(1,0,1)$-rows, the other $m_6 - (B - r_3)$ $(1,1,1)$-rows, all $(1,1,0)$-rows, and lastly all $(1,0,0)$-rows. In the resultant row permutation (see Figure 4(b)), all 1's in the first and the third columns are consecutive, and the second one of the two 1-bands in the second column has size $(q_3+1)B$. It is therefore an optimal solution. Symmetrically, if $m_6+r_5 > B$ or $m_6+r_7 > B$, we are also able to obtain an optimal row permutation achieving MAX bandpasses.

In the sequel, from $m_6+r_3 < B$ we conclude that $q_6 = 0$ (replacing m_6 by r_6). If $r_2+r_3+r_6 \geq B$ (Case 3.3), or equivalently $r_2 \geq B-r_3-r_6 > 0$, we stack in order $r_2-(B-r_3-r_6)$ $(0,0,1)$-rows, then all $(1,0,1)$-rows, all $(1,0,0)$-rows, all $(1,1,0)$-rows, all $(1,1,1)$-rows, all $(0,1,1)$-rows, and lastly the other $B - r_3 - r_6$ $(0,0,1)$-rows. In the resultant row permutation (see Figure 4(c)), all 1's in the first two columns are consecutive, and the first one of the two 1-bands in the third column has size

$(q_3+1)B$. Therefore, the row permutation is an optimal solution. Symmetrically, if $r_5+r_6+r_8 \geq B$, we are also able to obtain an optimal row permutation achieving MAX bandpasses.

In the sequel, we have $r_2+r_3+r_6 < B$, and thus $r_7 \geq B-r_2-r_3-r_6 > 0$ since $r_2+r_3+r_6+r_7 \geq B$. When $r_2+r_3+r_5+2r_6+r_7+r_8 \geq 2B$ (Case 3.4), we stack in order $m_7-(B-r_2-r_3-r_6)$ $(1,0,1)$-rows, then all $(1,0,0)$-rows, all $(1,1,0)$-rows, all $(1,1,1)$-rows, all $(0,1,1)$-rows, all $(0,0,1)$-rows, and the other $B-r_2-r_3-r_6$ $(1,0,1)$-rows. In the resultant row permutation (see Figure 4(d)), the second one of the two 1-bands in the first column has size $(q_5+q_7-1)B+r_2+r_3+r_5+2r_6+r_7+r_8$, achieving q_5+q_7+1 bandpasses. all 1's in the second column are consecutive, and the first one of the two 1-bands in the third column has size $(q_3+1)B$. It is therefore an optimal row permutation generating MAX bandpasses.

In the remaining scenario (Case 3.5), we have $q_6, q_2, q_8 = 0$, $r_6+r_7 < B$, $r_2+r_7 < B$, $r_8+r_7 < B$, and $r_2+r_3+r_5+2r_6+r_7+r_8 < 2B$ (which implies $r_2+r_3+r_6 < B$ and $r_5+r_6+r_8 < B$). Hence $MAX = 2(q_3+q_5+q_7)+3$. we convert all $(0,0,1)$-rows into $(0,1,1)$-rows, and convert all $(1,0,0)$-rows into $(1,1,0)$-rows, by adding 1's. This reduces the original instance I into a new instance I' such that $OPT(I) \leq OPT(I')$ and $MAX(I) = MAX(I')$.

We prove that $OPT(I') = MAX(I')-1$. Note that in instance I', $q'_3 = q_3$, $r'_3 = r_2+r_3$, $q'_5 = q_5$, $r'_5 = r_8+r_5$, $q'_7 = q_7$, $r'_7 = r_7$, and $r'_6 = r_6$. Assume to the contrary that the optimal row placement $\mathcal{P}'$ generates $MAX(I')$ bandpasses. We want to construct another new instance I'', which is initialized to be I', and one of its row placement $\mathcal{P}''$, which is initialized to be $\mathcal{P}'$. For each of the $r'_6 = r_6$ $(1,1,1)$-rows, if it participates in no bandpasses in $\mathcal{P}'$ across all three columns, then we remove it from I'' as well as $\mathcal{P}''$; if it participates in at most two bandpasses in $\mathcal{P}'$, assuming without loss of generality from the first two columns, then we replace it with a $(1,1,0)$-row in I'' as well as in $\mathcal{P}''$; if it participates in three bandpasses in $\mathcal{P}'$, assuming without loss of generality that among these three the top one is from the first column and the middle one is from the second column (and thus the bottom one is from the third column), then we accumulate all the $(1,1,1)$-rows that participate in these three bandpasses in $\mathcal{P}'$, and replace them with exactly the same number of $(1,1,0)$-rows and the same number of $(0,1,1)$-rows in I'', as well as in $\mathcal{P}''$ in which these $(1,1,0)$-rows are stacked on right top of these $(0,1,1)$-rows. At the end, the resultant new instance I'' has only three types of rows: m''_3 $(0,1,1)$-rows with $m'_3 \leq m''_3 \leq m'_3+r'_6$, m''_5 $(1,1,0)$-rows with $m'_5 \leq m''_5 \leq m'_5+r'_6$, and m''_7 $(1,0,1)$-rows with $m'_7 \leq m''_7 \leq m'_7+r'_6$. Furthermore, from the above row-replacing scheme, $m''_3+m''_5+m''_7 \leq m'_3+m'_5+2r'_6+m'_7$. It follows that instance I'' satisfies all the premises in Lemma 2, and consequently $OPT(I'') = MAX(I'')-1 = 2(q_3+q_5+q_7)+2$. However, the construction of row placement $\mathcal{P}''$ does not decrease the number of bandpasses generated in $\mathcal{P}'$. That is, $\mathcal{P}''$ is a solution to I'' with $MAX(I'') = 2(q_3+q_5+q_7)+3$ bandpasses, a contradiction. This contradiction proves that $OPT(I') = MAX(I')-1$. Therefore, $OPT(I) = MAX(I)-1$ too, and all its six row-stacking solutions are optimal.

5.1.2 When $q_4 > 0$

In this case, $m_4 = q_4B > 0$. Similarly as before, we only need to consider Case 3.5 for $q_4 = 0$ in the last section, where $q_6, q_2, q_8 = 0$, $r_6+r_7 < B$, $r_2+r_7 < B$, $r_8+r_7 < B$, and $r_2+r_3+r_5+2r_6+r_7+r_8 < 2B$. It follows that $MAX = 2(q_3+q_5+q_7)+q_4+3$.

When $r_3+r_6+r_7 \geq B$ (Case 4.1), we may simply ignore all the r_2 $(0,0,1)$-rows (by stacking them last), to stack in order $B-r_5$ $(0,1,0)$-rows, then all $(1,1,0)$-rows, all $(1,0,0)$-rows, all $(1,0,1)$-rows, all $(1,1,1)$-rows, all $(0,1,1)$-rows, the other $m_4-(B-r_5)$ $(0,1,0)$-rows, and lastly all $(0,0,1)$-rows. In the resultant row permutation (see Figure 5(a)), all 1's in the first column are consecutive, the second one of the two 1-bands in the second column has size $(q_5+1)B$, and the second one of the two 1-bands in the third column has size $(q_3+q_7)B+r_3+r_6+r_7$, achieving the maximum possible number of bandpasses. It is therefore an optimal row permutation. Symmetrically, if $r_5+r_6+r_7 \geq B$ or $r_5+r_7+r_8 \geq B$, we are also able to obtain an optimal row permutation achieving MAX bandpasses. In the sequel, we have $r_3+r_6+r_7 < B$, $r_5+r_6+r_7 < B$, and $r_5+r_7+r_8 < B$.

row type	quantity
$(0,0,1)$	r_2
$(0,1,0)$	$m_4-(B-r_5)$
$(0,1,1)$	m_3
$(1,1,1)$	r_6
$(1,0,1)$	m_7
$(1,0,0)$	r_8
$(1,1,0)$	m_5
$(0,1,0)$	$B-r_5$

(a) Case 4.1

row type	quantity
$(0,1,0)$	$m_4-(2B-r_2-r_3-r_5-r_6-r_7)$
$(1,1,1)$	$r_6-(B-r_5-r_7-r_8)$
$(0,1,1)$	m_3
$(0,0,1)$	r_2
$(1,1,1)$	$2B-(r_2+r_3+r_5+r_6+2r_7+r_8)$
$(1,0,1)$	m_7
$(1,0,0)$	r_8
$(1,1,0)$	m_5
$(1,1,1)$	$r_2+r_3+r_6+r_7-B$
$(0,1,0)$	$2B-r_2-r_3-r_5-r_6-r_7$

(c) Case 4.3

row type	quantity
$(0,1,0)$	$m_4-(r_7+r_8)$
$(1,1,1)$	$r_6-(B-r_5-r_7-r_8)$
$(0,1,1)$	m_3
$(0,0,1)$	r_2
$(1,0,1)$	m_7
$(1,0,0)$	r_8
$(1,1,0)$	m_5
$(1,1,1)$	$B-r_5-r_7-r_8$
$(0,1,0)$	r_7+r_8

(b) Case 4.2

row type	quantity
$(0,1,0)$	$m_4-(2B-r_2-r_3-r_5-r_6-r_7)$
$(1,1,1)$	$r_6-(B-r_5-r_7-r_8)$
$(0,1,1)$	$m_3-(r_2+r_3+r_5+r_6+2r_7+r_8-B)$
$(0,0,1)$	r_2
$(1,0,1)$	m_7
$(1,0,0)$	r_8
$(1,1,0)$	m_5
$(1,1,1)$	$B-r_5-r_7-r_8$
$(0,1,1)$	$r_2+r_3+r_5+r_6+2r_7+r_8-B$
$(0,1,0)$	$2B-r_2-r_3-r_5-r_6-r_7$

(d) Case 4.4

Figure 5: The optimal row placements when $r_6 > 0$ and $m_4 = q_4B > 0$.

Since $r_5+r_6+r_7+r_8 \geq B$, we have $r_6 \geq B-r_5-r_7-r_8 > 0$. When $r_2+r_3+r_5+r_6+2r_7+r_8 \geq 2B$ (Case 4.2), we stack in order r_7+r_8 $(0,1,0)$-rows, then $B-r_5-r_7-r_8$ $(1,1,1)$-rows, all $(1,1,0)$-rows, all $(1,0,0)$-rows, all $(1,0,1)$-rows, all $(0,0,1)$-rows, all $(0,1,1)$-rows, the other $r_6-(B-r_5-r_7-r_8)$ $(1,1,1)$-rows, and lastly the other $m_4-(r_7+r_8)$ $(0,1,0)$-rows. In the resultant row permutation (see Figure 5(b)), the second one of the two 1-bands in the first column has size $(q_5+q_7+1)B$, the second one of two 1-bands in the second column has size $(q_5+1)B$, and the first one of the two 1-bands in the third column has size $(q_3+q_7-1)B+r_2+r_3+r_5+r_6+2r_7+r_8$, achieving the maximum possible number of bandpasses. It is therefore an optimal solution of MAX bandpasses.

When $r_2 + r_3 + r_5 + r_6 + 2r_7 + r_8 < 2B$, we swap some $(0, 1, 0)$-rows on the top with the same number of $(1, 1, 1)$-rows from the bottom. This number is $2B - (r_2 + r_3 + r_5 + r_6 + 2r_7 + r_8)$. That is, if $r_2 + 2r_3 + 2r_5 + 2r_6 + 2r_7 + r_8 \geq 3B$ (Case 4.3, we stack in order $2B - r_2 - r_3 - r_5 - r_6 - r_7$ $(0, 1, 0)$-rows, then $r_2 + r_3 + r_6 + r_7 - B$ $(1, 1, 1)$-rows, all $(1, 1, 0)$-rows, all $(1, 0, 0)$-rows, all $(1, 0, 1)$-rows, $2B - (r_2 + r_3 + r_5 + r_6 + 2r_7 + r_8)$ $(1, 1, 1)$-rows, all $(0, 0, 1)$-rows, all $(0, 1, 1)$-rows, the other $r_6 - (B - r_5 - r_7 - r_8)$ $(1, 1, 1)$-rows, and lastly the other $m_4 - (2B - r_2 - r_3 - r_5 - r_6 - r_7)$ $(0, 1, 0)$-rows. In the resultant row permutation (see Figure 5(c)), the second one of the two 1-bands in the first column has size $(q_5 + q_7 + 1)B$, the first and the third of the three 1-bands in the second column have size $(q_3 + q_4)B + (r_2 + 2r_3 + 2r_5 + 2r_6 + 2r_7 + r_8 - 3B)$ and $(q_5 + 1)B$ respectively, achieving together the maximum possible number of bandpasses, and the first one of the two 1-bands in the third column has size $(q_3 + q_7 + 1)B$. It is therefore an optimal row permutation generating MAX bandpasses.

If $r_2 + 2r_3 + 2r_5 + 2r_6 + 2r_7 + r_8 < 3B$ but one of $\{q_3, q_5, q_7\}$ is positive, say $q_3 > 0$ (Case 4.4), then instead of using $(1, 1, 1)$-rows in Figure 5(c), we use $(0, 1, 1)$-rows to adjust for bandpasses. That is, we stack in order $2B - r_2 - r_3 - r_5 - r_6 - r_7$ $(0, 1, 0)$-rows, then $r_2 + r_3 + r_5 + r_6 + 2r_7 + r_8 - B$ $(0, 1, 1)$-rows, $B - r_5 - r_7 - r_8$ $(1, 1, 1)$-rows, all $(1, 1, 0)$-rows, all $(1, 0, 0)$-rows, all $(1, 0, 1)$-rows, all $(0, 0, 1)$-rows, the other $r_6 - (B - r_5 - r_7 - r_8)$ $(1, 1, 1)$-rows, the other $m_3 - (r_2 + r_3 + r_5 + r_6 + 2r_7 + r_8 - B)$ $(0, 1, 1)$-rows, and lastly the other $m_4 - (2B - r_2 - r_3 - r_5 - r_6 - r_7)$ $(0, 1, 0)$-rows. In this row placement (see Figure 5(d)), the second one of the two 1-bands in the first column has size $(q_5 + q_7 + 1)B$, the second one of the two 1-bands in the second column has size $(q_5 + 2)B$, and the first one of the two 1-bands in the third column has size $(q_3 + q_7)B$. Therefore, the row permutation is an optimal solution of MAX bandpasses.

In the remaining scenario (Case 4.5), that is, $q_2, q_3, q_5, q_6, q_7, q_8 = 0$ (and replace m_3, m_5, m_7 by r_3, r_5, r_7 respectively), $r_3 + r_6 + r_7 < B$, $r_5 + r_6 + r_7 < B$, $r_2 + r_3 + r_5 + 2r_6 + r_7 + r_8 < 2B$, $r_2 + r_3 + r_5 + r_6 + 2r_7 + r_8 < 2B$ (these last two imply that $r_2 + r_3 + r_6 < B$, $r_2 + r_3 + r_7 < B$, $r_8 + r_5 + r_6 < B$, and $r_8 + r_5 + r_7 < B$), $r_2 + 2r_3 + 2r_5 + 2r_6 + 2r_7 + r_8 < 3B$, and $m_4 = q_4 B > 0$, we have $MAX = 1 + (q_4 + 1) + 1 = q_4 + 3$. If there were an optimal row placement $\mathcal{P}$ achieving MAX bandpasses, then we conclude that the only 1-band in the first column and the only 1-band in the third column must overlap with at least $2B - (r_2 + r_3 + r_5 + r_6 + r_7 + r_8) > 0$ rows, since no $(0, 1, 0)$ is used for forming these two bandpasses. Furthermore, these overlapping rows are either $(1, 0, 1)$ or $(1, 1, 1)$, and none of them should be used for forming bandpasses in the second column in this placement $\mathcal{P}$, due to $r_3 + r_6 + r_7 < B$ and $r_5 + r_6 + r_7 < B$. This implies a total number of rows $\geq (q_4 + 1)B + 2B - (r_2 + r_3 + r_5 + r_6 + r_7 + r_8) + r_2 + r_8$, or equivalently $r_2 + 2r_3 + 2r_5 + 2r_6 + 2r_7 + r_8 \geq 3B$, a contradiction. That is, for this last scenario, $OPT = MAX - 1$ and thus all the six row-stacking solutions are optimal.

5.2 When $r_2 \cdot r_4 \cdot r_8 \neq 0$

In this case, none of r_2, r_4, r_8 is zero. We distinguish the following two subcases according to whether there is a zero in $\{r_3, r_5, r_7\}$.

5.2.1 When $r_3 \cdot r_5 \cdot r_7 = 0$

Assume without loss of generality that $r_5 = 0$. One can verify that Table 1 reduces to the following Table 4. We further separate into two disjoint subcases according to whether $q_5 = 0$.

Column permutations	Sizes of two 1-bands modulo B
$(1,2,3)$	$(r_2+r_3)\%B$, $(r_6+r_7)\%B$
$(2,1,3)$	$(r_2+r_7)\%B$, $(r_6+r_3)\%B$
$(1,3,2)$	$(r_4+r_3)\%B$, r_6
$(3,1,2)$	r_4, $(r_6+r_3)\%B$
$(2,3,1)$	$(r_8+r_7)\%B$, r_6
$(3,2,1)$	r_8, $(r_6+r_7)\%B$

Table 4: The sizes modulo B of the two 1-bands in the third column in the row-stacking solutions when $r_6 \cdot r_2 \cdot r_4 \cdot r_8 \neq 0$ and $r_5 = 0$.

5.1.2.1: When $q_5 = 0$.

If $m_3 + r_2 > B$ (Case 5.1), we stack in order all $(1,0,0)$-rows, then all $(1,0,1)$-rows, all $(1,1,1)$-rows, $m_3 - (B - r_2)$ $(0,1,1)$-rows, all $(0,1,0)$-rows, the other $B - r_2$ $(0,1,1)$-rows, and lastly all $(0,0,1)$-rows. In the resultant row permutation (see Figure 6(a)), all 1's in the first two columns are consecutive respectively, and the first one of the two 1-bands in the third column has size $(q_2+1)B$. It is therefore an optimal solution. Symmetrically, if $m_3 + r_4 > B$ or $m_3 + r_6 > B$ or $m_7 + r_2 > B$ or $m_7 + r_8 > B$ or $m_7 + r_6 > B$, we are also able to obtain an optimal row permutation achieving MAX bandpasses. So in the sequel, we have $q_3, q_7 = 0$, thus replacing m_3, m_7 by r_3, r_7 respectively, $r_3 + r_j < B$ for $j = 2, 4, 6$, and $r_7 + r_j < B$ for $j = 2, 8, 6$.

When $r_3 + r_6 + r_7 \geq B$ (Case 5.2), from $r_7 + r_6 < B$ we have $r_3 \geq B - r_6 - r_7 > 0$. We stack in order all $(1,0,0)$-rows, then all $(1,0,1)$-rows, all $(1,1,1)$-rows, $B - r_6 - r_7$ $(0,1,1)$-rows, all $(0,1,0)$-rows, the other $r_3 - (B - r_6 - r_7)$ $(0,1,1)$-rows, and lastly all $(0,0,1)$-rows. In the resultant row permutation (see Figure 6(b)), all 1's in the first two columns are consecutive respectively, and the second one of the two 1-bands in the third column has size $(q_6+1)B$. It is therefore an optimal solution. In the sequel we have $r_3 + r_6 + r_7 < B$.

When $r_2+r_3+r_6+r_7+r_8 \geq 2B$ (Case 5.3), we have $r_2+r_3+r_6 > B$ since $r_7+r_8 < B$, and thus $r_6 > B - r_2 - r_3 > 0$. We stack in order all $(1,0,0)$-rows, then all $(1,0,1)$-rows, $m_6 - (B - r_2 - r_3)$ $(1,1,1)$-rows, all $(0,1,0)$-rows, the other $B - r_2 - r_3$ $(1,1,1)$-rows, all $(0,1,1)$-rows, and lastly all $(0,0,1)$-rows. In the resultant row permutation (see Figure 6(c)), the second one of the two 1-bands in the first column has size $(q_6 + q_8 - 1)B + r_2 + r_3 + r_6 + r_7 + r_8$, achieving the maximum possible $q_6 + q_8 + 1$ bandpasses, all 1's in the second column are consecutive, and the first one of the two 1-bands in the third column has size $(q_2+1)B$. It is therefore an optimal solution. Symmetrically, if $r_4 + r_3 + r_6 + r_7 + r_8 \geq 2B$ or $r_2 + r_3 + r_6 + r_7 + r_4 \geq 2B$, we are also able to obtain an optimal row permutation achieving MAX bandpasses.

Suppose next $r_2+r_3+r_6+r_7+r_8 < 2B$, $r_2+r_3+r_6+r_7+r_4 < 2B$, and $r_4+r_3+r_6+r_7+r_8 < 2B$. When $r_2 + r_4 + r_8 + 2(r_3 + r_6 + r_7) \geq 3B$ and $q_2 > 0$ (Case 5.4, or $q_4 > 0$ or $q_8 > 0$, which can be analogously discussed), we stack in order $2B - r_3 - r_4 - r_6 - r_7$ $(0,0,1)$-rows, then all $(1,0,1)$-

row type	quantity
$(0,0,1)$	m_2
$(0,1,1)$	$B - r_2$
$(0,1,0)$	m_4
$(0,1,1)$	$m_3 - (B - r_2)$
$(1,1,1)$	m_6
$(1,0,1)$	m_7
$(1,0,0)$	m_8

(a) Case 5.1

row type	quantity
$(0,0,1)$	m_2
$(0,1,1)$	$r_3 - (B - r_6 - r_7)$
$(0,1,0)$	m_4
$(0,1,1)$	$B - r_6 - r_7$
$(1,1,1)$	m_6
$(1,0,1)$	r_7
$(1,0,0)$	m_8

(b) Case 5.2

row type	quantity
$(0,0,1)$	m_2
$(0,1,1)$	r_3
$(1,1,1)$	$B - r_2 - r_3$
$(0,1,0)$	m_4
$(1,1,1)$	$m_6 - (B - r_2 - r_3)$
$(1,0,1)$	r_7
$(1,0,0)$	m_8

(c) Case 5.3

row type	quantity
$(0,0,1)$	$m_2 - (2B - r_3 - r_4 - r_6 - r_7)$
$(0,1,1)$	r_3
$(1,1,1)$	$m_6 - B + r_7 + r_8$
$(0,1,0)$	m_4
$(1,1,1)$	$2B - r_3 - r_4 - r_6 - r_7 - r_8$
$(1,0,0)$	m_8
$(1,1,1)$	$r_3 + r_4 + r_6 - B$
$(1,0,1)$	r_7
$(0,0,1)$	$2B - r_3 - r_4 - r_6 - r_7$

(d) Case 5.4

Figure 6: The optimal row placements when $r_6 \cdot r_2 \cdot r_4 \cdot r_8 \neq 0$ and $m_5 = 0$.

rows, $r_3 + r_4 + r_6 - B$ $(1,1,1)$-rows, all $(1,0,0)$-rows, $2B - r_3 - r_4 - r_6 - r_7 - r_8$ $(1,1,1)$-rows, all $(0,1,0)$-rows, the other $m_6 - (r_3 + r_4 + r_6 - B) - (2B - r_3 - r_4 - r_6 - r_7 - r_8) = m_6 - B + r_7 + r_8$ $(1,1,1)$-rows, all $(0,1,1)$-rows, and lastly the other $m_2 - (2B - r_3 - r_4 - r_6 - r_7)$ $(0,0,1)$-rows. In the resultant row permutation (see Figure 6(d)), the second one of the two 1-bands in the first column has size $(q_8 + 1)B$, the first one of the two 1-bands in the second column has size $(q_4 + q_6 + 1)B$, and the first and the third of the three 1-bands in the third column have size $(q_2 + q_6 - 3)B + r_2 + r_4 + r_8 + 2(r_3 + r_6 + r_7)$ and B respectively, thus together achieving the maximum possible $q_2 + q_6 + 1$ bandpasses. It is therefore an optimal solution.

When $r_2+r_4+r_8+2(r_3+r_6+r_7) < 3B$ or when $r_2+r_4+r_8+2(r_3+r_6+r_7) \geq 3B$ but $q_2, q_4, q_8 = 0$ (Case 5.5), we convert all $(0,1,1)$-rows and all $(1,0,1)$-rows into $(1,1,1)$-rows by adding 1's. Since $r_3+r_6+r_7 < B$, $r_2+r_3+r_6+r_7+r_8 < 2B$, $r_4+r_3+r_6+r_7+r_8 < 2B$, and $r_4+r_3+r_6+r_7+r_2 < 2B$, this reduces the instance I into a new instance I', which satisfies the premises of Lemma 3. Since $OPT(I) \leq OPT(I') = MAX(I') - 1 = MAX(I) - 1$, we have $OPT(I) = MAX(I) - 1$, and thus all six row-stacking solutions to the original instance I are optimal.

5.1.2.2: When $q_5 > 0$.

We only need to consider Case 5.5 for $q_5 = 0$ in the last section, where $q_3, q_7 = 0$, $r_3 + r_6 + r_7 < B$, $r_2 + r_3 + r_6 + r_7 + r_8 < 2B$, $r_4 + r_3 + r_6 + r_7 + r_8 < 2B$, and $r_4 + r_3 + r_6 + r_7 + r_2 < 2B$ (Case

6.1). From $r_6 + r_7 + r_8 \geq B$, we conclude that $r_8 > r_3$. We stack in order r_3 $(1,0,0)$-rows, then $m_5 - (r_3 + r_6)$ $(1,1,0)$-rows, all $(1,1,1)$-rows, all $(0,1,1)$-rows, all $(0,0,1)$-rows, all $(1,0,1)$-rows, the other $m_8 - r_3$ $(1,0,0)$-rows, the other $r_3 + r_6$ $(1,1,0)$-rows, and lastly all $(0,1,0)$-rows. In the resultant row placement (see Figure 7), the second one of the two 1-bands in the first column has size $(q_5 + q_6)B$, the second one of the two 1-bands in the second column has size $(q_5 + q_6)B$, and all 1's in the third column are consecutive. It is therefore an optimal solution.

row type in order	quantity
$(0,1,0)$	m_4
$(1,1,0)$	$r_3 + r_6$
$(1,0,0)$	$m_8 - r_3$
$(1,0,1)$	r_7
$(0,0,1)$	m_2
$(0,1,1)$	r_3
$(1,1,1)$	m_6
$(1,1,0)$	$m_5 - (r_3 + r_6)$
$(1,0,0)$	r_3

Figure 7: The optimal row placements when $r_6 \cdot r_2 \cdot r_4 \cdot r_8 \neq 0$ and $m_5 = q_5 B > 0$.

5.2.2 When $r_3 \cdot r_5 \cdot r_7 \neq 0$

That is, for all $i = 2, 3, \ldots, 8$, $r_i > 0$. We separate two scenarios according to whether $q_6 = 0$.

5.2.2.1: When $q_6 = 0$.

In this section, we replace m_6 by r_6. Consider first when $r_5 + r_6 > B$, that is, $m_5 \geq r_5 > B - r_6 > 0$. If $m_8 + r_7 \geq B$ (Case 7.1), then we stack in order $m_8 - (B - r_7)$ $(1,0,0)$-rows, then $m_5 - (B - r_6)$ $(1,1,0)$-rows, all $(0,1,0)$-rows, the other $B - r_6$ $(1,1,0)$-rows, all $(1,1,1)$-rows, all $(0,1,1)$-rows, all $(0,0,1)$-rows, all $(1,0,1)$-rows, and lastly the other $B - r_7$ $(1,0,0)$-rows. In the resultant row permutation (see Figure 8(a)), the first and the second of the three 1-bands in the first column have size $q_7 B$ and B respectively, and all 1's in the second and the third columns are consecutive, respectively. It is therefore an optimal solution. Symmetrically, if $m_2 + r_7 \geq B$ or $m_2 + r_3 \geq B$ or $m_4 + r_3 \geq B$, we are also able to achieve an optimal row placement with MAX bandpasses. In the sequel we deal with the remaining case where $q_2, q_4, q_8 = 0$ (thus replacing m_2, m_4, m_8 by r_2, r_4, r_8 respectively), and $r_8 + r_7 < B$, $r_2 + r_7 < B$, $r_2 + r_3 < B$, $r_4 + r_3 < B$.

From $r_5 + r_6 > B$, $r_8 + r_7 < B$, and $(r_5 + r_6)\%B + (r_7 + r_8)\%B \geq B$, we conclude that $2B \leq r_5 + r_6 + r_7 + r_8 < 3B$ and thus $B < r_6 + r_7 + r_8 < 2B$. It follows that $m_5 \geq r_5 \geq 2B - r_6 - r_7 - r_8 > 0$ and $r_6 > B - r_7 - r_8 > 0$. If $m_3 \geq r_8$ (Case 7.2), then we stack in order all $(0,0,1)$-rows, then $m_3 - r_8$ $(0,1,1)$-rows, $r_6 - (B - r_7 - r_8)$ $(1,1,1)$-rows, all $(1,1,0)$-rows, all $(0,1,0)$-rows, the other r_8 $(0,1,1)$-rows, the other $B - r_7 - r_8$ $(1,1,1)$-rows, all $(1,0,1)$-rows, and lastly all $(1,0,0)$-rows. In the resultant row permutation (see Figure 8(b)), the first one of the two 1-bands in the first column has size $(q_7 + 1)B$, all 1's in the second column are consecutive, and the first one of the two 1-bands in the third column has size $(q_7 + 1)B$. It is therefore an optimal solution. Symmetrically, if $m_7 \geq r_4$, we are also able to obtain an optimal row permutation achieving MAX bandpasses. So in the sequel we further assume that $q_3, q_7 = 0$ (thus replacing m_3, m_7 by r_3, r_7 respectively), $r_3 < r_8$, and $r_7 < r_4$.

row type	quantity
$(1,0,0)$	$B - r_7$
$(1,0,1)$	m_7
$(0,0,1)$	m_2
$(0,1,1)$	m_3
$(1,1,1)$	r_6
$(1,1,0)$	$B - r_6$
$(0,1,0)$	m_4
$(1,1,0)$	$m_5 - (B - r_6)$
$(1,0,0)$	$m_8 - (B - r_7)$

(a) Case 7.1

row type	quantity
$(1,0,0)$	r_8
$(1,0,1)$	m_7
$(1,1,1)$	$B - r_7 - r_8$
$(0,1,1)$	r_8
$(0,1,0)$	r_4
$(1,1,0)$	m_5
$(1,1,1)$	$r_6 - (B - r_7 - r_8)$
$(0,1,1)$	$m_3 - r_8$
$(0,0,1)$	r_2

(b) Case 7.2

row type	quantity
$(1,0,0)$	r_8
$(1,0,1)$	r_7
$(1,1,1)$	$B - r_3 - r_7$
$(0,1,1)$	r_3
$(0,1,0)$	r_4
$(0,0,1)$	r_2
$(1,1,1)$	$r_6 - (B - r_3 - r_7)$
$(1,1,0)$	r_5

(c) Case 7.3

row type	quantity
$(1,0,0)$	r_8
$(1,0,1)$	r_7
$(1,1,1)$	$r_6 - (B - r_2 - r_3)$
$(1,1,0)$	r_5
$(0,1,0)$	r_4
$(1,1,1)$	$B - r_2 - r_3$
$(0,1,1)$	r_3
$(0,0,1)$	r_2

(d) Case 7.4

row type	quantity
$(1,0,0)$	r_8
$(1,1,0)$	$m_5 - (B - r_6 - r_7)$
$(0,1,0)$	$r_4 - r_7$
$(0,1,1)$	r_3
$(0,0,1)$	r_2
$(1,0,1)$	r_7
$(1,1,1)$	r_6
$(1,1,0)$	$B - r_6 - r_7$
$(0,1,0)$	r_7

(e) Case 7.6

Figure 8: The optimal row placements when $\Pi_{i=2}^{8} r_i \neq 0$, $q_6 = 0$, and $r_5 + r_6 > B$.

Symmetric to the above discussion, if $r_7 + r_6 > B$, we may further assume that $r_8 + r_5 < B$, $r_4 + r_5 < B$, $q_5 = 0$ (thus replacing m_5 by r_5), and $r_5 < r_2$, since otherwise we are able to analogously obtain an optimal row permutation achieving MAX bandpasses. Next, if $r_3 + r_5 + r_7 + r_6 \geq 2B$ (Case 7.3), then we conclude that $r_3 + r_7 + r_6 > B$, $r_5 + r_6 + r_7 + r_8 \geq 2B - r_3 + r_8 > 2B$, $r_3 + r_4 + r_5 + r_6 \geq 2B - r_7 + r_4 > 2B$, and $r_2 + r_3 + r_6 + r_7 \geq 2B - r_5 + r_2 > 2B$. Therefore, $MAX = 2 + 2 + 2 = 6$. It also follows that $r_6 > B - r_3 - r_7 > B - r_8 - r_7 > 0$. We stack in order all $(1,1,0)$-rows, then $r_6 - (B - r_3 - r_7)$ $(1,1,1)$-rows, all $(0,0,1)$-rows, all $(0,1,0)$-rows, all $(0,1,1)$-rows, the other $B - r_3 - r_7$ $(1,1,1)$-rows, all $(1,0,1)$-rows, and lastly all $(1,0,0)$-rows. In the resultant row permutation (see Figure 8(c)), the first one of the two 1-bands in the first column has size $B + r_8 - r_3 > B$ and the second one has size $r_3 + r_5 + r_6 + r_7 - B > B$, thus achieving two bandpasses, the first one of the two 1-bands in the second column has size $B + r_4 - r_7 > B$, thus achieving one bandpass, and the first one of the two 1-bands in the third column has size B. It is therefore an optimal solution generating $MAX = 6$ bandpasses.

In the sequel we have $r_3 + r_5 + r_7 + r_6 < 2B$. From $r_2 + r_3 < B$ and $r_2 + r_3 + r_6 > B$ we conclude that $r_6 > B - r_2 - r_3 > 0$. Hence, if $r_2 + r_3 + r_5 + r_7 + r_6 + r_8 \geq 3B$ (Case 7.4), we stack in order all $(0,0,1)$-rows, then all $(0,1,1)$-rows, $B - r_2 - r_3$ $(1,1,1)$-rows, all $(0,1,0)$-rows, all $(1,1,0)$-rows, the other $r_6 - (B - r_2 - r_3)$ $(1,1,1)$-rows, all $(1,0,1)$-rows, and lastly all $(1,0,0)$-rows. In the resultant row permutation (see Figure 8(d)), the first one of the two 1-bands in the first column has size $r_2 + r_3 + r_5 + r_6 + r_7 + r_8 - B \geq 2B$, all the 1's in the second column are consecutive, and the second one of the two 1-bands in the third column has size B. It is therefore an optimal solution generating $MAX = 6$ bandpasses. Symmetrically, if $r_2 + r_3 + r_5 + r_6 + r_7 + r_4 \geq 3B$ or $r_4 + r_3 + r_5 + r_6 + r_7 + r_8 \geq 3B$, we are also able to achieve an optimal row placement with $MAX = 6$ bandpasses.

In the sequel (Case 7.5), we have $q_2, q_4, q_8, q_3, q_5, q_7 = 0$, $r_8 + r_7 < B$, $r_2 + r_7 < B$, $r_8 + r_5 < B$, $r_4 + r_5 < B$, $r_2 + r_3 < B$, $r_4 + r_3 < B$, $r_7 < r_4$, $r_3 < r_8$, $r_5 < r_2$, $r_3 + r_5 + r_7 + r_6 < 2B$, $r_2 + r_3 + r_5 + r_7 + r_6 + r_8 < 3B$, $r_4 + r_3 + r_5 + r_7 + r_6 + r_2 < 3B$, and $r_8 + r_3 + r_5 + r_7 + r_6 + r_4 < 3B$. We convert all $(0,1,1)$-, $(1,1,0)$-, and $(1,0,1)$-rows into $(1,1,1)$-rows by adding 1's, to reduce to a new instance I'. Clearly, $OPT(I) \leq OPT(I')$. Instance I' contains only four types of rows, with $r'_i = r_i$ for $i = 2,4,8$ and $r'_6 = r_3 + r_5 + r_7 + r_6 - B$. It therefore satisfies the premises described in Lemma 3, and thus $OPT(I') = MAX(I') - 1 = MAX(I) - 1$. This shows that $OPT(I) = MAX(I) - 1$, and therefore all six row-stacking solutions are all optimal.

If $r_7 + r_6 < B$ (Case 7.6), then we have $B < r_5 + r_6 + r_7 < 2B$ and thus $m_5 \geq r_5 > B - r_6 - r_7 > 0$. Recall that we have $r_7 < r_4$. We stack in order r_7 $(0,1,0)$-rows, then $B - r_6 - r_7$ $(1,1,0)$-rows, all $(1,1,1)$-rows, all $(1,0,1)$-rows, all $(0,0,1)$-rows, all $(0,1,1)$-rows, the other $r_4 - r_7$ $(0,1,0)$-rows, the other $m_5 - (B - r_6 - r_7)$ $(1,1,0)$-rows, and lastly all $(1,0,0)$-rows. In the resultant row permutation (see Figure 8(e)), the second one of the two 1-bands in the first column has size B, the second one of the two 1-bands in the second column has size B too, and all the 1's in the third column are consecutive. It is therefore an optimal solution. This finishes up the discussion for the case when $r_5 + r_6 > B$.

It the rest of this section, we consider when $r_5 + r_6 < B$. In fact, the above discussion tells that we only need to consider when $r_i + r_6 < B$ for all $i = 3, 5, 7$. It follows that Table 1 reduces to the following Table 5. Apparently, if $r_2 + r_3 > B$, then $r_2 + r_3 + r_7 + r_6 \geq 2B$ and thus $r_2 + r_7 > B$; For the same reason, $r_2 + r_7 > B$ implies $r_2 + r_3 > B$; and analogously, $r_4 + r_3 > B$ if and only if $r_4 + r_5 > B$, and $r_8 + r_7 > B$ if and only if $r_8 + r_5 > B$.

Consider first when $m_2 + r_3 > B$, which implies $m_2 + r_7 > B$ from the last paragraph. If $m_7 + r_8 > B$ (Case 7.7), then we stack in order all $(1,0,0)$-rows, then $B - r_8$ $(1,0,1)$-rows,

Column permutations	Sizes of two 1-bands modulo B
$(1,2,3)$	$(r_2+r_3)\%B,\ r_7+r_6$
$(2,1,3)$	$(r_2+r_7)\%B,\ r_3+r_6$
$(1,3,2)$	$(r_4+r_3)\%B,\ r_5+r_6$
$(3,1,2)$	$(r_4+r_5)\%B,\ r_3+r_6$
$(2,3,1)$	$(r_8+r_7)\%B,\ r_5+r_6$
$(3,2,1)$	$(r_8+r_5)\%B,\ r_7+r_6$

Table 5: The sizes modulo B of the two 1-bands in the third column in the row-stacking solutions when $\Pi_{i=2}^{8} r_i \neq 0$, $q_6 = 0$, and $r_i + r_6 < B$ for $i = 3, 5, 7$.

$m_2 - (B - r_3)$ $(0,0,1)$-rows, the other $m_7 - (B - r_8)$ $(1,0,1)$-rows, all $(1,1,1)$-rows, all $(1,1,0)$-rows, all $(0,1,0)$-rows, all $(0,1,1)$-rows, and lastly the other $B - r_3$ $(0,0,1)$-rows. In the resultant row permutation (see Figure 9(a)), the second one of the two 1-bands in the first column has size $(q_8+1)B$, all 1's in the second column are consecutive, and the first one of the two 1-bands in the third column has size $(q_3+1)B$. It is therefore an optimal solution. Symmetrically, if $m_3 + r_4 > B$, we are also able to achieve an optimal row placement with MAX bandpasses. In the sequel, we have $q_3, q_7 = 0$ (thus replacing m_3, m_7 by r_3, r_7 respectively), $r_8 + r_7 < B$, and $r_4 + r_3 < B$ (and consequently $r_8 + r_5 < B$, $r_4 + r_5 < B$).

If $q_5 > 0$ (Case 7.8), and assuming $r_4 \geq r_8$ (the opposite case can be analogously discussed), we have $m_5 > B - r_8$ and stack in order $m_2 - (B - r_3)$ $(0,0,1)$-rows, then all $(1,0,1)$-rows, all $(1,1,1)$-rows, $m_5 - (B - r_8)$ $(1,1,0)$-rows, $m_4 - r_8$ $(0,1,0)$-rows, all $(0,1,1)$-rows, the other $B - r_3$ $(0,0,1)$-rows, all $(1,0,0)$-rows, the other $B - r_8$ $(1,1,0)$-rows, and lastly the other r_8 $(0,1,0)$-rows. In the resultant row permutation (see Figure 9(b)), the first one of the two 1-bands in the first column has size $(q_8+1)B$, the first one of the two 1-bands in the second column has size B, and the first one of the two 1-bands in the third column has size B. It is therefore an optimal solution. In the following we have $q_5 = 0$, and replace m_5 by r_5.

Next, when $r_5 + r_6 + r_7 \geq B$ (Case 7.9), we stack in order all $(1,0,0)$-rows, then $m_2 - (B - r_3)$ $(0,0,1)$-rows, all $(1,0,1)$-rows, all $(1,1,1)$-rows, all $(1,1,0)$-rows, all $(0,1,0)$-rows, all $(0,1,1)$-rows, and lastly the other $B - r_3$ $(0,0,1)$-rows. In the resultant row permutation (see Figure 9(c)), the first column achieves $1 + q_8$ bandpasses, all 1's in the second column are consecutive, and the first one of the two 1-bands in the third column has size B. It is therefore an optimal solution. In Case 7.9, essentially the remainder r_8 $(1,0,0)$-rows are not used for forming bandpasses. Symmetrically, if $r_3 + r_4 + r_5 \geq B$ or $r_3 + r_5 + r_6 \geq B$ or $r_5 + r_7 + r_8 \geq B$, we are also able to obtain an optimal row permutation achieving MAX bandpasses. Therefore, we have in the sequel $r_5 + r_6 + r_7 < B$, $r_3 + r_4 + r_5 < B$, $r_3 + r_5 + r_6 < B$, and $r_5 + r_7 + r_8 < B$.

Recall that we are considering the case of $m_2 + r_3 > B$, which contains two possible subcases: $r_2 + r_3 > B$, and $r_2 + r_3 < B$ but $q_2 > 0$. When $r_2 + r_3 > B$ (implying $r_2 + r_7 > B$), we have $r_7 > B - r_2$ and $r_2 + r_3 + r_6 + r_7 \geq 2B$. It follows that $MAX = (q_8 + 1) + (q_4 + 1) + (q_2 + 2)$.

If $r_3 + r_4 + r_6 \geq B$ (Case 7.10), we stack in order all $(0,1,0)$-rows, then all $(0,1,1)$-rows, all $(1,1,1)$-rows, $r_7 - (B - r_2)$ $(1,0,1)$-rows, all $(1,1,0)$-rows, all $(1,0,0)$-rows, the other $B - r_2$ $(1,0,1)$-rows, and lastly all $(0,0,1)$-rows. In the resultant row permutation (see Figure 9(d)), all 1's in the first column are consecutive, the second one of the two 1-bands in the second column

row type	quantity
$(0,0,1)$	$B - r_3$
$(0,1,1)$	m_3
$(0,1,0)$	m_4
$(1,1,0)$	m_5
$(1,1,1)$	r_6
$(1,0,1)$	$m_7 - (B - r_8)$
$(0,0,1)$	$m_2 - (B - r_3)$
$(1,0,1)$	$B - r_8$
$(1,0,0)$	m_8

(a) Case 7.7

row type	quantity
$(0,1,0)$	r_8
$(1,1,0)$	$B - r_8$
$(1,0,0)$	m_8
$(0,0,1)$	$B - r_3$
$(0,1,1)$	r_3
$(0,1,0)$	$m_4 - r_8$
$(1,1,0)$	$m_5 - (B - r_8)$
$(1,1,1)$	r_6
$(1,0,1)$	r_7
$(0,0,1)$	$m_2 - (B - r_3)$

(b) Case 7.8

row type	quantity
$(0,0,1)$	$B - r_3$
$(0,1,1)$	r_3
$(0,1,0)$	m_4
$(1,1,0)$	r_5
$(1,1,1)$	r_6
$(1,0,1)$	r_7
$(0,0,1)$	$m_2 - (B - r_3)$
$(1,0,0)$	m_8

(c) Case 7.9

row type	quantity
$(0,0,1)$	m_2
$(1,0,1)$	$B - r_2$
$(1,0,0)$	m_8
$(1,1,0)$	r_5
$(1,0,1)$	$r_7 - (B - r_2)$
$(1,1,1)$	r_6
$(0,1,1)$	r_3
$(0,1,0)$	m_4

(d) Case 7.10

row type	quantity
$(0,0,1)$	$B - r_3$
$(0,1,1)$	r_3
$(0,1,0)$	m_4
$(1,1,1)$	r_6
$(1,1,0)$	r_5
$(1,0,0)$	m_8
$(1,0,1)$	r_7
$(0,0,1)$	$m_2 - (B - r_3)$

(e) Case 7.11

row type	quantity
$(0,0,1)$	$B - r_3$
$(0,1,1)$	r_3
$(0,1,0)$	m_4
$(1,1,1)$	$r_6 - (2B - r_2 - r_3 - r_7)$
$(1,1,0)$	r_5
$(1,0,0)$	m_8
$(1,0,1)$	r_7
$(1,1,1)$	$2B - r_2 - r_3 - r_7$
$(0,0,1)$	$m_2 - (B - r_3)$

(f) Case 7.12

Figure 9: The optimal row placements when $\Pi_{i=2}^{8} r_i \neq 0$, $q_6 = 0$, $r_i + r_6 < B$ for $i = 3, 5, 7$, and $m_2 + r_3 > B$.

has size $q_4B + r_3 + r_4 + r_6 \geq (q_4 + 1)B$, achieving $q_4 + 1$ bandpasses, and the first one of the two 1-bands in the third column has size $(q_2 + 1)B$. It is therefore an optimal solution. In Case 7.10, essentially those $(1,1,0)$-rows are not used for forming bandpasses in the second column. Symmetrically, if $r_6 + r_7 + r_8 \geq B$, we are also able to obtain an optimal row permutation achieving MAX bandpasses. We have in the sequel $r_3 + r_4 + r_6 < B$ and $r_6 + r_7 + r_8 < B$.

If $r_2 + r_3 + r_7 \geq 2B$ (Case 7.11), we stack in order $m_2 - (B - r_3)$ $(0,0,1)$-rows, then all $(1,0,1)$-rows, all $(1,0,0)$-rows, all $(1,1,0)$-rows, all $(1,1,1)$-rows, all $(0,1,0)$-rows, all $(0,1,1)$-rows, and lastly the other $B - r_3$ $(0,0,1)$-rows. In the resultant row permutation (see Figure 9(e)), all 1's in the first two columns are consecutive respectively, and the first and the third of the three 1-bands in the third column have size B and $(q_2 - 1)B + r_2 + r_3 + r_7 \geq (q_2 + 1)B$. It is therefore an optimal solution.

When $r_2 + r_3 + r_7 < 2B$, we conclude that $r_6 > 2B - r_2 - r_3 - r_7 > 0$. If $r_2 + 2r_3 + r_4 + r_5 + r_6 + r_7 \geq 3B$ (Case 7.12), we stack in order $m_2 - (B - r_3)$ $(0,0,1)$-rows, then $2B - r_2 - r_3 - r_7$ $(1,1,1)$-rows, all $(1,0,1)$-rows, all $(1,0,0)$-rows, all $(1,1,0)$-rows, the other $r_6 - (2B - r_2 - r_3 - r_7)$ $(1,1,1)$-

rows, all $(0,1,0)$-rows, all $(0,1,1)$-rows, and lastly the other $B-r_3$ $(0,0,1)$-rows. In the resultant row permutation (see Figure 9(f)), all 1's in the first column are consecutive, the second column achieves q_4+1 bandpasses, and the first and the third of the three 1-bands in the third column have size B and $(q_2+1)B$ respectively. It is therefore an optimal solution. Symmetrically, if $r_2+r_3+r_5+r_6+2r_7+r_8 \geq 3B$, we are also able to obtain an optimal row permutation achieving MAX bandpasses.

Next, from $r_2+r_3+r_6+r_7 \geq 2B$ we conclude that $r_3+r_6+r_7 > B$, and henceforth $r_3 > B-r_6-r_7 > 0$. So when $q_4 > 0$ (Case 7.13), we stack in order $B-r_5$ $(0,1,0)$-rows, then all $(1,1,0)$-rows, all $(1,0,0)$-rows, all $(1,0,1)$-rows, all $(1,1,1)$-rows, $B-r_6-r_7$ $(0,1,1)$-rows, the other $m_4-(B-r_5)$ $(0,1,0)$-rows, the other $r_3-(B-r_6-r_7)$ $(0,1,1)$-rows, and lastly all $(0,0,1)$-rows. In the resultant row permutation (see Figure 10(a)), all 1's in the first column are consecutive, the second one of the two 1-bands in the second column has size B, and the second one of the two 1-bands in the third column has size B too. It is therefore an optimal solution. Symmetrically, if $q_8 > 0$, we are also able to obtain an optimal row permutation achieving MAX bandpasses. Therefore, we further assume that in the sequel $q_4, q_8 = 0$, and replace m_4, m_8 by r_4, r_8 respectively. It follows that $MAX = 1+1+(q_2+2)$.

From $r_3+r_4+r_5+r_6 \geq B$ we conclude that $r_5 > B-r_3-r_4-r_6 > 0$. So if $r_3+r_4+r_5+2r_6+r_7+r_8 \geq 2B$ (Case 7.14), we stack in order $r_5-(B-r_3-r_4-r_6)$ $(1,1,0)$-rows, then all $(1,0,0)$-rows, all $(1,0,1)$-rows, all $(1,1,1)$-rows, $B-r_6-r_7$ $(0,1,1)$-rows, all $(0,1,0)$-rows, the other $B-r_3-r_4-r_6$ $(1,1,0)$-rows, the other $r_3-(B-r_6-r_7)$ $(0,1,1)$-rows, and lastly all $(0,0,1)$-rows. In the resultant row permutation (see Figure 10(b)), the second one of the two 1-bands in the first column has size at least B, achieving one bandpass, the first one of the two 1-bands in the second column has size exactly B, and the second one of the two 1-bands in the third column has size B too. It is therefore an optimal solution.

From $r_2+r_3+r_6+r_7 \geq 2B$, $r_5+r_6+r_7+r_8 \geq B$, and $r_2+r_3+r_5+r_6+2r_7+r_8 < 3B$, we have $r_6 = (r_2+r_3+r_6+r_7-2B)+(r_5+r_6+r_7+r_8-B)+(3B-r_2-r_3-r_5-r_6-2r_7-r_8)$, and $r_2+r_3+r_5+r_6+r_7+r_8-2B < B-r_7 < r_2$. Hence if $r_2+r_4+r_8+2r_3+2r_5+2r_7+2r_6 \geq 4B$ (Case 7.15), we stack in order $r_2+r_3+r_5+r_6+r_7+r_8-2B$ $(0,0,1)$-rows, then all $(1,0,1)$-rows, $3B-r_2-r_3-r_5-r_6-2r_7-r_8$ $(1,1,1)$-rows, all $(1,0,0)$-rows, all $(1,1,0)$-rows, $r_2+r_3+r_6+r_7-2B$ $(1,1,1)$-rows, all $(0,1,0)$-rows, all $(0,1,1)$-rows, the other $r_5+r_6+r_7+r_8-B$ $(1,1,1)$-rows, and lastly the other $m_2-(r_2+r_3+r_5+r_6+r_7+r_8-2B)$ $(0,0,1)$-rows. In the resultant row permutation (see Figure 10(c)), the second one of the two 1-bands in the first column has size exactly B, the first one of the two 1-bands in the second column has size at least B, achieving one bandpass, and the first and the third of the three 1-bands in the third column have size $(q_2+1)B$ and B respectively. It is therefore an optimal solution.

In the other scenario of $r_2+r_3 > B$ (Case 7.16), assuming that $OPT = MAX = 1+1+(q_2+2)$, there are one bandpass in the first column and one bandpass in the second column of the optimal row permutation $\mathcal{P}$. Since $(0,0,1)$-rows are not used in these two bandpasses, we conclude that these two bandpasses share at least $2B-r_3-r_4-r_5-r_6-r_7-r_8 > r_6$ rows, which must contain at least one $(1,1,0)$-row and some $(1,1,1)$-rows. Firstly, none of these shared $(1,1,1)$-rows will be used for forming bandpasses in the third column, since otherwise we would have either $r_3+r_5+r_6 \geq B$ or $r_5+r_6+r_7 \geq B$. Secondly, as there must be q_2+2 bandpasses in the third column, and because $(0,1,0)$- and $(1,0,0)$-rows are not used for forming any of them, the total number of rows is at least $(q_2+2)B+(2B-r_3-r_4-r_5-r_6-r_7-r_8)+r_4+r_8 = (q_2+4)B-r_3-r_5-r_6-r_7$, which is

row type	quantity
$(0,0,1)$	m_2
$(0,1,1)$	$r_3-(B-r_6-r_7)$
$(0,1,0)$	$m_4-(B-r_5)$
$(0,1,1)$	$B-r_6-r_7$
$(1,1,1)$	r_6
$(1,0,1)$	r_7
$(1,0,0)$	m_8
$(1,1,0)$	r_5
$(0,1,0)$	$B-r_5$

(a) Case 7.13

row type	quantity
$(0,0,1)$	m_2
$(0,1,1)$	$r_3-(B-r_6-r_7)$
$(1,1,0)$	$B-r_3-r_4-r_6$
$(0,1,0)$	r_4
$(0,1,1)$	$B-r_6-r_7$
$(1,1,1)$	r_6
$(1,0,1)$	r_7
$(1,0,0)$	r_8
$(1,1,0)$	$r_5-(B-r_3-r_4-r_6)$

(b) Case 7.14

row type	quantity
$(0,0,1)$	$m_2-(r_2+r_3+r_5+r_6+r_7+r_8-2B)$
$(1,1,1)$	$r_5+r_6+r_7+r_8-B$
$(0,1,1)$	r_3
$(0,1,0)$	r_4
$(1,1,1)$	$r_2+r_3+r_6+r_7-2B$
$(1,1,0)$	r_5
$(1,0,0)$	r_8
$(1,1,1)$	$3B-r_2-r_3-r_5-r_6-2r_7-r_8$
$(1,0,1)$	r_7
$(0,0,1)$	$r_2+r_3+r_5+r_6+r_7+r_8-2B$

(c) Case 7.15

Figure 10: The optimal row placements when $\Pi_{i=2}^{8} r_i \neq 0$, $q_6 = 0$, $r_i + r_6 < B$ for $i = 3, 5, 7$, and $m_2 + r_3 > B$.

strictly greater than $q_2 B + r_2 + r_3 + r_4 + r_5 + r_6 + r_7 + r_8$, a contradiction. Such a contradiction shows that in this last scenario, $OPT = MAX - 1$ and thus any of the six row-stacking solutions is optimal.

When $r_2 + r_3 < B$ (and thus $r_2 + r_7 < B$) but $q_2 > 0$ (they together still guarantee that $m_2 + r_3 > B$), the discussion is merged with the case where $m_2 + r_3 < B$, implying $q_2 = 0$. Since the above discussion for $m_2 + r_3 > B$, Cases 7.7–7.16, only misses out the situation where $r_2 + r_3 < B$ and $q_2 > 0$, we only need to cover the *merged* case where $r_2 + r_3 < B$, $r_4 + r_3 < B$, $r_8 + r_7 < B$. Furthermore, if one of $\{q_2, q_4, q_8\}$ is positive, without loss of generality $q_2 > 0$, then $q_3, q_5, q_7 = 0$ (see Cases 7.7 and 7.8), $r_5 + r_6 + r_7 < B$, $r_3 + r_4 + r_5 < B$, $r_3 + r_5 + r_6 < B$, and $r_5 + r_7 + r_8 < B$ (see Case 7.9). Note that in this case Table 5 reduces to the following Table 6.

Consider first some of $\{q_3, q_5, q_7\}$ is positive, and thus there must be $q_2, q_4, q_8 = 0$ (replacing m_2, m_4, m_8 by r_2, r_4, r_8 respectively) and $MAX = (q_5 + q_7 + 1) + (q_3 + q_5 + 1) + (q_3 + q_7 + 1)$. Without loss of generality, assume $q_7 > 0$. If $r_6 + r_7 + r_8 \geq B$ (Case 7.17), we stack in order all $(1,1,0)$-rows, then all $(0,1,0)$-rows, all $(0,1,1)$-rows, all $(1,1,1)$-rows, $m_7 - (B - r_2)$ $(1,0,1)$-

Column permutations	Sizes of two 1-bands modulo B
$(1, 2, 3)$	$r_2 + r_3, \ \ r_7 + r_6$
$(2, 1, 3)$	$r_2 + r_7, \ \ r_3 + r_6$
$(1, 3, 2)$	$r_4 + r_3, \ \ r_5 + r_6$
$(3, 1, 2)$	$r_4 + r_5, \ \ r_3 + r_6$
$(2, 3, 1)$	$r_8 + r_7, \ \ r_5 + r_6$
$(3, 2, 1)$	$r_8 + r_5, \ \ r_7 + r_6$

Table 6: The sizes modulo B of the two 1-bands in the third column in the row-stacking solutions when $\Pi_{i=2}^{8} r_i \neq 0$, $q_6 = 0$, $r_i + r_6 < B$ for $i = 3, 5, 7$, $r_2 + r_3 < B$, $r_4 + r_3 < B$, and $r_8 + r_7 < B$.

rows, all $(1, 0, 0)$-rows, the other $B - r_2$ $(1, 0, 1)$-rows, and lastly all $(0, 0, 1)$-rows. In the resultant row permutation (see Figure 11(a)), the first one of the two 1-bands in the first column has size $q_7 B + r_6 + r_7 + r_8 \geq (q_7 + 1)B$, all 1's in the second column are consecutive, and the first one of the two 1-bands in the third column has size B. It is therefore an optimal solution.

If $r_4 + r_5 + r_6 \geq B$ (Case 7.18), we stack in order all $(0, 1, 0)$-rows, then all $(1, 1, 0)$-rows, all $(1, 1, 1)$-rows, $m_7 - (B - r_8)$ $(1, 0, 1)$-rows, all $(0, 1, 1)$-rows, all $(0, 0, 1)$-rows, the other $B - r_8$ $(1, 0, 1)$-rows, and lastly all $(1, 0, 0)$-rows. In the resultant row permutation (see Figure 11(b)), the first one of the two 1-bands in the first column has size B, the second one of the two 1-bands in the second column has size $q_5 B + r_4 + r_5 + r_6 \geq (q_5 + 1)B$, and all 1's in the third column are consecutive. It is therefore an optimal solution. Therefore, we assume that in the sequel $r_6 + r_7 + r_8 < B$ and $r_4 + r_5 + r_6 < B$.

If $r_2 + r_6 + r_7 + r_8 \geq B$ (Case 7.19), then from $r_6 + r_7 + r_8 < B$ we have $r_2 \geq B - r_6 - r_7 - r_8 > 0$. We stack in order $B - r_6 - r_7 - r_8$ $(0, 0, 1)$-rows, then $m_7 - (B - r_8)$ $(1, 0, 1)$-rows, all $(1, 1, 1)$-rows, all $(1, 1, 0)$-rows, all $(0, 1, 0)$-rows, all $(0, 1, 1)$-rows, the other $r_2 - (B - r_6 - r_7 - r_8)$ $(0, 0, 1)$-rows, and lastly all $(1, 0, 0)$-rows. In the resultant row permutation (see Figure 11(c)), the first one of the two 1-bands in the first column has size B, all 1's in the second column are consecutive, and the second one of the two 1-bands in the third column has size $q_7 B$. It is therefore an optimal solution, and we assume in the following that $r_2 + r_6 + r_7 + r_8 < B$.

It follows from $r_2 + r_3 + r_6 + r_7 \geq B$ that $r_3 > B - r_2 - r_6 - r_7 - r_8 > 0$. If $r_2 + r_3 + r_4 + r_5 + 2r_6 + r_7 + r_8 \geq 2B$ (Case 7.20), we stack in order all $(0, 0, 1)$-rows, then $B - r_2 - r_6 - r_7 - r_8$ $(0, 1, 1)$-rows, $m_7 - (B - r_8)$ $(1, 0, 1)$-rows, all $(1, 1, 1)$-rows, all $(1, 1, 0)$-rows, all $(0, 1, 0)$-rows, the other $m_3 - (B - r_2 - r_6 - r_7 - r_8)$ $(0, 1, 1)$-rows, the other $B - r_8$ $(1, 0, 1)$-rows, and lastly all $(1, 0, 0)$-rows. In the resultant row permutation (see Figure 11(d)), the first one of the two 1-bands in the first column has size B, the first one of the two 1-bands in the second column has size $(q_3 + q_5 - 1)B + r_2 + r_3 + r_4 + r_5 + 2r_6 + r_7 + r_8 \geq (q_3 + q_5 + 1)B$, and the second one of the two 1-bands in the third column has size $q_7 B$. It is therefore an optimal solution.

For the remaining scenario (Case 7.21), we prove similarly as in Case 3.5 that $OPT = MAX - 1$. Assume to the contrary that the optimal row placement $\mathcal{P}$ generates MAX bandpasses. We want to construct a new instance I', which is initialized to be the original instance (denoted as I for ease of presentation), and one of its row placement $\mathcal{P}'$, which is initialized to be $\mathcal{P}$. For each of the r_6 $(1, 1, 1)$-rows, if it participates in no bandpasses in $\mathcal{P}$ across all three columns, then we remove it from I' as well as $\mathcal{P}'$; if it participates in at most two bandpasses in $\mathcal{P}$, assuming without loss of

row type	quantity
$(0,0,1)$	r_2
$(1,0,1)$	$B - r_2$
$(1,0,0)$	r_8
$(1,0,1)$	$m_7 - (B - r_2)$
$(1,1,1)$	r_6
$(0,1,1)$	m_3
$(0,1,0)$	r_4
$(1,1,0)$	m_5

(a) Case 7.17

row type	quantity
$(1,0,0)$	r_8
$(1,0,1)$	$B - r_8$
$(0,0,1)$	r_2
$(0,1,1)$	m_3
$(1,0,1)$	$m_7 - (B - r_8)$
$(1,1,1)$	r_6
$(1,1,0)$	m_5
$(0,1,0)$	r_4

(b) Case 7.18

row type	quantity
$(1,0,0)$	r_8
$(1,0,1)$	$B - r_8$
$(0,0,1)$	$r_2 - (B - r_6 - r_7 - r_8)$
$(0,1,1)$	m_3
$(0,1,0)$	r_4
$(1,1,0)$	m_5
$(1,1,1)$	r_6
$(1,0,1)$	$m_7 - (B - r_8)$
$(0,0,1)$	$B - r_6 - r_7 - r_8$

(c) Case 7.19

row type	quantity
$(1,0,0)$	r_8
$(1,0,1)$	$B - r_8$
$(0,1,1)$	$m_3 - (B - r_2 - r_6 - r_7 - r_8)$
$(0,1,0)$	r_4
$(1,1,0)$	m_5
$(1,1,1)$	r_6
$(1,0,1)$	$m_7 - (B - r_8)$
$(0,1,1)$	$B - r_2 - r_6 - r_7 - r_8$
$(0,0,1)$	r_2

(d) Case 7.20

Figure 11: The optimal row placements when $\Pi_{i=2}^{8} r_i \neq 0$, $q_6 = 0$, $r_i + r_6 < B$ for $i = 3, 5, 7$, $r_2 + r_3 < B$, $r_4 + r_3 < B$, $r_8 + r_7 < B$, and $q_7 > 0$.

generality from the first two columns, then we replace it with a $(1,1,0)$-row in I' as well as in $\mathcal{P}'$; if it participates in three bandpasses in $\mathcal{P}$, assuming without loss of generality that among these three the top one is from the second column and the middle one is from the third column, then we accumulate all the $(1,1,1)$-rows that participate in these three bandpasses in $\mathcal{P}$, and replace them with exactly the same number of $(0,1,1)$-rows and the same number of $(1,0,1)$-rows in I', as well as in $\mathcal{P}'$ in which these $(0,1,1)$-rows are stacked on right top of these $(1,0,1)$-rows. Next, we convert all $(0,0,1)$- and $(1,0,0)$-rows to $(1,0,1)$-rows, and convert all $(0,1,0)$-rows to $(1,1,0)$-rows in I', by adding 1's. At the end, the resultant new instance I' has only three types of rows: m'_3 $(0,1,1)$-rows with $q'_3 = q_3$ and $r_3 \le r'_3 \le r_3 + r_6 < B$, m'_5 $(1,1,0)$-rows with $q'_5 = q_5$ and $r_5 \le r'_5 \le r_5 + r_4 + r_6 < B$, and m'_7 $(1,0,1)$-rows with $q'_7 = q_7$ and $r_7 \le r'_7 \le r_7 + r_2 + r_6 + r_8 < B$. Furthermore, from the above row-replacing scheme, $r'_3 + r'_5 + r'_7 \le r_2 + r_3 + r_4 + r_5 + 2r_6 + r_7 + r_8 < 2B$. It follows that instance I' satisfies all the premises in Lemma 2, and consequently $OPT(I') = MAX(I') - 1 = 2(q_3 + q_5 + q_7) + 2$. However, the construction of row placement $\mathcal{P}'$ does not decrease the number of bandpasses generated in $\mathcal{P}$. That is, $\mathcal{P}'$ is a solution to I' with $MAX = 2(q_3 + q_5 + q_7) + 3$ bandpasses, a contradiction. This contradiction proves that $OPT = MAX - 1$, and therefore all the six row-stacking solutions are optimal.

We consider next the case where $q_3, q_5, q_7 = 0$, and we replace m_3, m_5, m_7 by r_3, r_5, r_7 respectively. It follows that $MAX = (q_8 + 1) + (q_4 + 1) + (q_2 + 1)$. Since $r_4 + r_5 < B$, we denote $x = B - r_4 - r_5 > 0$. If $(r_2 + r_3 + r_6 + r_7) + (r_4 + r_5) \geq 2B$ (Case 7.22), then $r_2 + r_3 + r_6 + r_7 \geq B + x$. When $r_6 \geq x$, we stack in order all $(0,0,1)$-rows, then all $(0,1,1)$-rows, $r_6 - x$ $(1,1,1)$-rows, all $(1,0,1)$-rows, all $(1,0,0)$-rows, all $(1,1,0)$-rows, the other x $(1,1,1)$-rows, and lastly all $(0,1,0)$-rows. The resultant row permutation is shown in Figure 12(a). When $r_6 < x$, then we conclude from $r_2 + r_7 < B$ that $r_3 + r_6 > x$, and we stack in order all $(0,0,1)$-rows, then $r_3 - (x - r_6)$ $(0,1,1)$-rows, all $(1,0,1)$-rows, all $(1,0,0)$-rows, all $(1,1,0)$-rows, all $(1,1,1)$-rows, the other $x - r_6$ $(0,1,1)$-rows, and lastly all $(0,1,0)$-rows. The resultant row permutation is shown in Figure 12(b). In both resultant row permutations, all 1's in the first column are consecutive, the first one of the two 1-bands in the second column has size $(q_4 + 1)B$, and the second one of the two 1-bands in the third column has size $q_2 B + r_2 + r_3 + r_6 + r_7 - x \geq (q_2 + 1)B$. They are therefore optimal. Symmetrically, if $(r_5 + r_6 + r_7 + r_8) + (r_2 + r_3) \geq 2B$ or $(r_3 + r_4 + r_5 + r_6) + (r_8 + r_7) \geq 2B$, we are also able to obtain an optimal row permutation achieving MAX bandpasses. So, in the sequel we have $r_2 + r_3 + r_4 + r_5 + r_6 + r_7 + r_8 - r_i < 2B$, for $i = 2, 4, 8$.

When $q_2, q_4, q_8 = 0$ (replacing m_2, m_4, m_8 by r_2, r_4, r_8 respectively), if $r_3 + r_6 + r_7 \geq B$ and $r_3 + r_4 + r_5 + 2r_6 + r_7 + r_8 \geq 2B$ (Case 7.23), we distinguish two scenarios. In the first scenario, $r_6 + r_7 + r_8 \geq B$, and we stack in order all $(0,0,1)$-rows, then all $(0,1,0)$-rows, all $(1,1,0)$-rows, all $(0,1,1)$-rows, all $(1,1,1)$-rows, all $(1,0,1)$-rows, and lastly all $(1,0,0)$-rows. In the resultant row permutation (see Figure 12(c)), the first one of the two 1-bands in the first column has size $r_6 + r_7 + r_8 \geq B$, thus achieving one bandpass, all 1's in the second column are consecutive, and the first one of the two 1-bands in the third column has size $r_3 + r_6 + r_7 \geq B$, thus achieving one bandpass. It is therefore an optimal solution achieving $MAX = 3$ bandpasses. In the second scenario, $r_6 + r_7 + r_8 < B$, we have $r_5 \geq B - r_6 - r_7 - r_8 > 0$, and we stack in order all $(0,0,1)$-rows, then all $(0,1,0)$-rows, $r_5 - (B - r_6 - r_7 - r_8)$ $(1,1,0)$-rows, all $(0,1,1)$-rows, all $(1,1,1)$-rows, all $(1,0,1)$-rows, all $(1,0,0)$-rows, and lastly the other $B - r_6 - r_7 - r_8$ $(1,1,0)$-rows. In the resultant row permutation (see Figure 12(d)), the first one of the two 1-bands in the first column has size B, the second one of the two 1-bands in the second column has size $r_3 + r_4 + r_5 + 2r_6 + r_7 + r_8 - B \geq B$, thus achieving one bandpass, and the first one of the two 1-bands in the third column have size $r_3 + r_6 + r_7 \geq B$, thus achieving one bandpass. It is therefore an optimal solution achieving $MAX = 3$ bandpasses. Symmetrically, if $r_3 + r_6 + r_5 \geq B$ and $r_2 + r_3 + r_5 + 2r_6 + r_7 + r_8 \geq 2B$, or $r_5 + r_6 + r_7 \geq B$ and $r_2 + r_3 + r_4 + r_5 + 2r_6 + r_7 \geq 2B$, we are also able to obtain an optimal row permutation achieving $MAX = 3$ bandpasses.

In the remaining scenario of $q_2, q_4, q_8 = 0$ (Case 7.24), we prove by contradiction that $OPT = 2$ and thus all six row-stacking solutions are optimal. Suppose otherwise there is an optimal row permutation $\mathcal{P}$ generating $MAX = 3$ bandpasses, one in each column. Recall that we have $r_2 + r_3 + r_4 + r_5 + r_6 + r_7 + r_8 - r_i < 2B$, for $i = 2, 4, 8$. Since $(0,0,1)$-rows are not used for forming bandpasses in the first and the second columns, these two bandpasses must share at least $2B - r_3 - r_4 - r_5 - r_6 - r_7 - r_8 > 0$ rows, which include some $(1,1,0)$-rows and some $(1,1,1)$-rows. It follows that these three bandpasses must be pairwise overlapping, and thus one of them uses all the rows shared by the other two. Assume without loss of generality that the rows shared by the two bandpasses in the first and the second columns are used by the bandpass in the third column (the other two cases can be analogously discussed). We conclude that $r_5 + r_6 + r_7 \geq B$ and the rows shared by the two bandpasses in the first and the second columns are all $(1,1,1)$-rows. Therefore, $2B - r_3 - r_4 - r_5 - r_6 - r_7 - r_8 \leq r_6$, for otherwise there would not be sufficient $(1,1,1)$-rows. That is, we have $r_5 + r_6 + r_7 \geq B$ and $r_2 + r_3 + r_4 + r_5 + 2r_6 + r_7 \geq 2B$, a contradiction.

row type	quantity
$(0,1,0)$	m_4
$(1,1,1)$	x
$(1,1,0)$	r_5
$(1,0,0)$	m_8
$(1,0,1)$	r_7
$(1,1,1)$	$r_6 - x$
$(0,1,1)$	r_3
$(0,0,1)$	m_2

(a) Case 7.22, $r_6 \geq x$

row type	quantity
$(0,1,0)$	m_4
$(0,1,1)$	$x - r_6$
$(1,1,1)$	r_6
$(1,1,0)$	r_5
$(1,0,0)$	m_8
$(1,0,1)$	r_7
$(0,1,1)$	$r_3 - (x - r_6)$
$(0,0,1)$	m_2

(b) Case 7.22, $r_6 < x$

row type	quantity
$(1,0,0)$	r_8
$(1,0,1)$	r_7
$(1,1,1)$	r_6
$(0,1,1)$	r_3
$(1,1,0)$	r_5
$(0,1,0)$	r_4
$(0,0,1)$	r_2

(c) Case 7.23, $r_6 + r_7 + r_8 \geq B$

row type	quantity
$(1,1,0)$	$B - r_6 - r_7 - r_8$
$(1,0,0)$	r_8
$(1,0,1)$	r_7
$(1,1,1)$	r_6
$(0,1,1)$	r_3
$(1,1,0)$	$r_5 - (B - r_6 - r_7 - r_8)$
$(0,1,0)$	r_4
$(0,0,1)$	r_2

(d) Case 7.23, $r_6 + r_7 + r_8 < B$

row type	quantity
$(0,0,1)$	$m_2 - (r_5 + r_8)$
$(1,1,1)$	$r_6 - (B - r_5 - r_7 - r_8)$
$(0,1,1)$	r_3
$(0,1,0)$	m_4
$(1,1,0)$	r_5
$(1,0,0)$	m_8
$(1,0,1)$	r_7
$(1,1,1)$	$B - r_5 - r_7 - r_8$
$(0,0,1)$	$r_5 + r_8$

(e) Case 7.25

row type	quantity
$(0,0,1)$	$m_2 - (2B - r_3 - r_4 - r_5 - r_6 - r_7)$
$(1,1,1)$	$r_5 + r_6 + r_7 + r_8 - B$
$(0,1,1)$	r_3
$(0,1,0)$	m_4
$(1,1,0)$	r_5
$(1,1,1)$	$2B - r_3 - r_4 - 2r_5 - r_6 - r_7 - r_8$
$(1,0,0)$	m_8
$(1,0,1)$	r_7
$(1,1,1)$	$r_3 + r_4 + r_5 + r_6 - B$
$(0,0,1)$	$2B - r_3 - r_4 - r_5 - r_6 - r_7$

(f) Case 7.26

Figure 12: The optimal row placements when $\Pi_{i=2}^{8} r_i \neq 0$, $q_6 = 0$, $r_i + r_6 < B$ for $i = 3, 5, 7$, $r_2 + r_3 < B$, $r_4 + r_3 < B$, $r_8 + r_7 < B$, and $q_3, q_5, q_7 = 0$.

When not all of q_2, q_4, q_8 are zero, we assume without loss of generality that $q_2 > 0$. It follows from Case 7.9 that we only need to consider the case where $r_3 + r_4 + r_5 < B$, $r_3 + r_5 + r_6 < B$, $r_5 + r_6 + r_7 < B$, and $r_5 + r_7 + r_8 < B$.

From $r_5 + r_7 + r_8 < B$, we have $r_6 > B - r_5 - r_7 - r_8 > 0$. So, if $r_3 + r_4 + 2r_5 + r_6 + r_7 + r_8 \geq 2B$ (Case

7.25), we stack in order r_5+r_8 $(0,0,1)$-rows, then $B-r_5-r_7-r_8$ $(1,1,1)$-rows, all $(1,0,1)$-rows, all $(1,0,0)$-rows, all $(1,1,0)$-rows, all $(0,1,0)$-rows, all $(0,1,1)$-rows, the other $r_6-(B-r_5-r_7-r_8)$ $(1,1,1)$-rows, and lastly the other $m_2-(r_5+r_8)$ $(0,0,1)$-rows. In the resultant row permutation (see Figure 12(e)), the second one of the two 1-bands in the first column has size $(q_8+1)B$, the first one of the two 1-bands in the second column has size $r_3+m_4+2r_5+r_6+r_7+r_8-B \geq (q_4+1)B$, and the second one of the two 1-bands in the third column has size B. It is therefore an optimal solution. So, in the sequel we have $r_3+r_4+2r_5+r_6+r_7+r_8 < 2B$, from which we have $r_6 \geq 2B-r_3-r_4-2r_5-r_6-r_7-r_8 > 0$.

Clearly, $r_6 = (r_3+r_4+r_5+r_6-B)+(r_5+r_6+r_7+r_8-B)+(2B-r_3-r_4-2r_5-r_6-r_7-r_8)$. If $r_2+r_4+r_8+2r_3+2r_5+2r_6+2r_7 \geq 3B$ (Case 7.26), then we stack in order $2B-r_3-r_4-r_5-r_6-r_7$ $(0,0,1)$-rows, then $r_3+r_4+r_5+r_6-B$ $(1,1,1)$-rows, all $(1,0,1)$-rows, all $(1,0,0)$-rows, the other $2B-r_3-r_4-2r_5-r_6-r_7-r_8$ $(1,1,1)$-rows, all $(1,1,0)$-rows, all $(0,1,0)$-rows, all $(0,1,1)$-rows, the other $r_5+r_6+r_7+r_8-B$ $(1,1,1)$-rows, and lastly the other $m_2-(2B-r_3-r_4-r_5-r_6-r_7)$ $(0,0,1)$-rows. In the resultant row permutation (see Figure 12(f)), the second one of the two 1-bands in the first column has size $(q_8+1)B$, the first one of the two 1-bands in the second column has size $(q_4+1)B$, and the first and the third of the three 1-bands in the third column have size $m_2+r_4+r_8+2r_3+2r_5+2r_6+2r_7-3B \geq q_2B$ and B respectively. It is therefore an optimal solution achieving MAX bandpasses. So, in the sequel we consider the case of $r_2+r_4+r_8+2r_3+2r_5+2r_6+2r_7 < 3B$.

Symmetric to the discussion of $q_2 > 0$ in Case 7.9, if $q_4 > 0$ or $q_8 > 0$, then an optimal row permutation with MAX bandpasses can be achieved when $r_3+r_6+r_7 \geq B$. When $r_3+r_6+r_7 < B$ (Case 7.27), we prove by contradiction that $OPT = MAX - 1$ and thus all six row-stacking solutions are optimal. Suppose otherwise there is an optimal row permutation $\mathcal{P}$ generating MAX bandpasses. Since $(0,0,1)$-rows are not used for forming bandpasses in the first and the second columns, these bandpasses must share at least $2B-r_3-r_4-r_5-r_6-r_7-r_8 > r_5$ rows, which are $(1,1,0)$- and $(1,1,1)$-rows. The bandpasses in the third column of $\mathcal{P}$ should not share any one of these, for otherwise it implies that either $r_3+r_5+r_6 \geq B$, or $r_5+r_6+r_7 \geq B$, or $r_3+r_6+r_7 \geq B$. Yet there are q_2+1 bandpasses in the third column, which do not use those shared rows, neither the $(0,1,0)$- or $(1,0,0)$-rows. Consequently, the total number of rows must be at least $(q_2+1)B+(2B-r_3-r_4-r_5-r_6-r_7-r_8)+m_4+m_8$, implying that $r_2+r_4+r_8+2r_3+2r_5+2r_6+2r_7 \geq 3B$, a contradiction.

In the remaining case, we have $q_4, q_8 = 0$ and replace m_4, m_8 by r_4, r_8 respectively. We also have $r_3+r_6+r_7 \geq B$ since otherwise we have proven that $OPT = MAX - 1$ in Case 7.27. From Case 7.23, when $r_3+r_4+r_5+2r_6+r_7+r_8 \geq 2B$, we are able to achieve an optimal solution generating MAX bandpasses. When $r_3+r_4+r_5+2r_6+r_7+r_8 < 2B$ (Case 7.28), we prove by contradiction similarly as in Case 7.16 that $OPT = MAX - 1$ and thus all six row-stacking solutions are optimal. Suppose otherwise there is an optimal row permutation $\mathcal{P}$ generating MAX bandpasses. Since $(0,0,1)$-rows are not used for forming bandpasses in the first and the second columns, these two bandpasses must share at least $2B-r_3-r_4-r_5-r_6-r_7-r_8 > r_6$ rows, which include at least one $(1,1,0)$-row and some $(1,1,1)$-rows. The bandpasses in the third column of $\mathcal{P}$ should not share any one of these, for otherwise it implies that either $r_3+r_5+r_6 \geq B$ or $r_5+r_6+r_7 \geq B$. Yet there are q_2+1 bandpasses in the third column, which do not use those shared rows, neither the $(0,1,0)$- or $(1,0,0)$-rows. Consequently, the total number of rows must be at least $(q_2+1)B+(2B-r_3-r_4-r_5-r_6-r_7-r_8)+r_4+r_8$, implying that $r_2+r_4+r_8+2r_3+2r_5+2r_6+2r_7 \geq 3B$, a contradiction. This finishes the discussion of $q_6 = 0$.

5.2.2.1: When $q_6 > 0$.

Note that we only need to deal with those cases among Cases 7.1-7.28 for which $OPT = MAX - 1$. These are Cases 7.5, 7.16, 7.21, 7.24, 7.27, and 7.28. Since $m_6 = q_6B + r_6 > B$, we might be able to achieve optimal row placements with MAX bandpasses. We thus follow the same discussion route when $q_6 = 0$.

In Case 7.5, we have $q_2, q_4, q_8, q_3, q_5, q_7 = 0$, $r_8 + r_7 < B$, $r_2 + r_7 < B$, $r_8 + r_5 < B$, $r_4 + r_5 < B$, $r_2+r_3 < B$, $r_4+r_3 < B$, $r_7 < r_4$, $r_3 < r_8$, $r_5 < r_2$, $r_3+r_5+r_7+r_6 < 2B$, $r_2+r_3+r_5+r_7+r_6+r_8 < 3B$, $r_4 + r_3 + r_5 + r_7 + r_6 + r_2 < 3B$, and $r_8 + r_3 + r_5 + r_7 + r_6 + r_4 < 3B$. So even with $q_6 > 0$ (Case 8.1), we still convert all $(0, 1, 1)$-, $(1, 1, 0)$-, and $(1, 0, 1)$-rows into $(1, 1, 1)$-rows by adding 1's, to reduce to a new instance I'. Clearly, $OPT(I) \leq OPT(I')$. Instance I' contains only four types of rows, with $r'_i = r_i$ for $i = 2, 4, 8$ and $r'_6 = r_3 + r_5 + r_7 + r_6 - B$. It therefore satisfies the premises described in Lemma 3, and thus $OPT(I') = MAX(I') - 1 = MAX(I) - 1$. This shows that $OPT(I) = MAX(I) - 1$, and therefore all six row-stacking solutions are all optimal.

In Case 7.16, we have $r_7 + r_6 < B$ and $r_2 + r_3 > B$, among other constraints. It follows from $r_2 + r_3 + r_6 + r_7 \geq 2B$ that $r_3 + r_6 + r_7 > B$, and thus $m_3 \geq r_3 > B - r_6 - r_7 > 0$. With $q_6 > 0$ (Case 8.2), we stack in order all $(1, 0, 0)$-rows, then all $(1, 0, 1)$-rows, $B - r_7$ $(1, 1, 1)$-rows, all $(1, 1, 0)$-rows, the other $m_6 - (B - r_7)$ $(1, 1, 1)$-rows, $B - r_6 - r_7$ $(0, 1, 1)$-rows, all $(0, 1, 0)$-rows, the other $m_3 - (B - r_6 - r_7)$ $(0, 1, 1)$-rows, and lastly all $(0, 0, 1)$-rows. In the resultant row permutation (see Figure 13(a)), all 1's in the first two columns are consecutive respectively, and the second and the third of the three 1-bands in the third column have size q_6B and $(q_7 + 1)B$ respectively. It is therefore an optimal solution with MAX bandpasses.

In Case 7.21, we have $q_7 > 0$ and $r_4 + r_3 < B$, among other constraints. With $q_6 > 0$ (Case 8.3), we stack in order all $(0, 0, 1)$-rows, then $m_7 - r_4$ $(1, 0, 1)$-rows, $m_6 - (B - r_3 - r_4)$ $(1, 1, 1)$-rows, all $(1, 1, 0)$-rows, all $(1, 0, 0)$-rows, the other r_4 $(1, 0, 1)$-rows, the other $B - r_3 - r_4$ $(1, 1, 1)$-rows, all $(0, 1, 1)$-rows, and lastly all $(0, 1, 0)$-rows. In the resultant row permutation (see Figure 13(b)), all 1's in the first column are consecutive, the first one of the two 1-bands in the second column has size $(q_3 + q_4 + 1)B$, and the first one of the two 1-bands in the third column has size $(q_3 + 1)B$, It is therefore an optimal solution with MAX bandpasses.

We merge the discussion of Cases 7.24, 7.27, and 7.28 by appending $q_6 > 0$ to the end of Case 7.22, where Cases 7.24, 7.27, and 7.28 stem from. That is, we have $q_3, q_5, q_7 = 0$, $r_2 + r_3 + r_4 + r_5 + r_6 + r_7 + r_8 - r_i < 2B$ for $i = 2, 4, 8$, and now $q_6 > 0$. It follows that $MAX = (q_8 + q_6 + 1) + (q_4 + q_6 + 1) + (q_2 + q_6 + 1)$.

If $r_7 \geq r_4$ (Case 8.4), the row placement in Figure 13(b) for Case 8.3 is feasible and also optimal in this case. Symmetrically, if $r_3 \geq r_8$ or $r_5 \geq r_2$, we are also able to achieve an optimal row placement with MAX bandpasses. In the sequel, we consider the case in which $r_7 < r_4$, $r_3 < r_8$, and $r_5 < r_2$.

If $r_3 + r_6 + r_7 \geq B$ (Case 8.5), then $r_3 + r_4 + r_6 > B$, and we stack in order all $(0, 0, 1)$-rows, then all $(1, 1, 0)$-rows, all $(1, 0, 0)$-rows, all $(1, 0, 1)$-rows, all $(1, 1, 1)$-rows, all $(0, 1, 1)$-rows, and lastly all $(0, 1, 0)$-rows. In the resultant row permutation (see Figure 13(c)), all the 1's in the first column are consecutive, the first one of the two 1-bands in the second column has size

row type	quantity
$(0,0,1)$	m_2
$(0,1,1)$	$m_3-(B-r_6-r_7)$
$(0,1,0)$	m_4
$(0,1,1)$	$B-r_6-r_7$
$(1,1,1)$	$m_6-(B-r_7)$
$(1,1,0)$	m_5
$(1,1,1)$	$B-r_7$
$(1,0,1)$	m_7
$(1,0,0)$	m_8

(a) Case 8.2

row type	quantity
$(0,1,0)$	m_4
$(0,1,1)$	m_3
$(1,1,1)$	$B-r_3-r_4$
$(1,0,1)$	r_4
$(1,0,0)$	m_8
$(1,1,0)$	m_5
$(1,1,1)$	$m_6-(B-r_3-r_4)$
$(1,0,1)$	m_7-r_4
$(0,0,1)$	m_2

(b) Cases 8.3 and 8.4

row type	quantity
$(0,1,0)$	m_4
$(0,1,1)$	r_3
$(1,1,1)$	m_6
$(1,0,1)$	r_7
$(1,0,0)$	m_8
$(1,1,0)$	r_5
$(0,0,1)$	m_2

(c) Case 8.5

row type in order	quantity
$(0,0,1)$	m_2
$(1,0,1)$	r_7
$(1,1,1)$	r_3+r_6
$(1,1,0)$	$B-r_3-r_7-r_6$
$(0,1,0)$	m_4
$(1,1,0)$	$r_3+r_5+r_7+r_6-B$
$(1,0,0)$	m_8
$(1,1,1)$	$m_6-(r_3+r_6)$
$(0,1,1)$	r_3

(d) Case 8.6

Figure 13: The optimal row placements when $\Pi_{i=2}^{8} r_i \neq 0$, $q_6 > 0$, and $r_i + r_6 < B$ for $i = 3, 5, 7$.

$(q_4+q_6)B+r_3+r_4+r_6 \geq (q_4+q_6+1)B$, and the first one of the two 1-bands in the third column has size $q_6B + r_3 + r_6 + r_7 \geq (q_6+1)B$. It is therefore an optimal solution. Symmetrically, if $r_3 + r_5 + r_6 \geq B$ or $r_5 + r_6 + r_7 \geq B$, we are also able to achieve an optimal row placement with MAX bandpasses. In the sequel, we consider the case in which $r_3 + r_6 + r_7 < B$, $r_3 + r_5 + r_6 < B$, and $r_5 + r_6 + r_7 < B$.

It follows from $r_3 + r_6 + r_7 < B$ that $r_5 > r_3 + r_5 + r_6 + r_7 - B$. So, if $r_3 + r_5 + r_7 + r_6 \geq B$ (Case 8.6), we stack in order all $(0,1,1)$-rows, then $m_6 - (r_3 + r_6)$ $(1,1,1)$-rows, all $(1,0,0)$-rows, $r_3+r_5+r_6+r_7-B$ $(1,1,0)$-rows, all $(0,1,0)$-rows, the other $r_5-(r_3+r_5+r_6+r_7-B) = B-r_3-r_6-r_7$ $(1,1,0)$-rows, the other $r_3 + r_6$ $(1,1,1)$-rows, all $(1,0,1)$-rows, and lastly all $(0,0,1)$-rows. In the resultant row permutation (see Figure 13(d)), the first one of the two 1-bands in the first column has size B, the second one of the two 1-bands in the second column has size q_6B, and the second one of the two 1-bands in the third column has size q_6B. It is therefore an optimal solution. We assume in the sequel $r_3 + r_5 + r_7 + r_6 < B$.

In the remaining scenario (Case 8.7), we convert all $(0,1,1)$-, $(1,1,0)$-, and $(1,0,1)$-rows into $(1,1,1)$-rows by adding 1's, to reduce to a new instance I'. Clearly, $OPT(I) \leq OPT(I')$. Instance I' contains only four types of rows, with $r'_i = r_i$ for $i = 2,4,8$ and $r'_6 = r_3 + r_5 + r_7 + r_6$. When

$q_2, q_4, q_8 = 0$ (as in Case 7.24), instance I' satisfies the premises described in Lemma 3, and thus $OPT(I') = MAX(I') - 1 = MAX(I) - 1$. When not all of q_2, q_4, q_8 are zero (as in Cases 7.27 and 7.28), we have $r_2 + r_4 + r_8 + 2r_3 + 2r_5 + 2r_6 + 2r_7 < 3B$ from the discussion of Case 7.26. Instance I' again satisfies the premises described in Lemma 3, and thus $OPT(I') = MAX(I') - 1 = MAX(I) - 1$. Therefore, we always have $OPT(I) = MAX(I) - 1$, suggesting that all six row-stacking solutions are optimal.

6 Conclusions

Theorem 4 *The three column Bandpass problem with any bandpass number $B \geq 2$ can be solved exactly in linear time.*

PROOF. In the last four sections we show that in most cases, the six solutions returned from the row-stacking algorithm include an optimal one; all the exceptional cases are recognized in Sections 4 and 5, for each of which an optimal row permutation generating MAX bandpasses is constructed in linear time, while all six row-stacking solutions generate only $MAX - 1$ bandpasses. $\square$

The algorithm solving the three column Bandpass problem has been implemented into a JAVA program, which is available upon request.

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