Representing the vertices of a graph with subtrees of a tree

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Set Representations

Let:
$S = \{s_1 \ldots s_n\}$ be a family of sets
$G = (V, E)$ be a graph
$\phi$ be some set relationship

then

$G$ is the $\phi$-graph of $S$ iff for $1 \leq i, j \leq n$
holds $v_i, v_j$ are adjacent iff $s_i, s_j$ are $\phi$
Overlap Representation

\{ \{2, 3, 4\}, \{2, 3\}, \{1, 2\}, \{5, 3\} \}
Overlap Graphs of Subtrees of a Tree

Graph $G = (V, E)$ is a **subtree overlap graph** (SOG) iff there exists a family of subtrees $\mathcal{T}$ of tree $T$ such that $G$ is the overlap graph of $\mathcal{T}$.
Intersection Graphs of Subtrees of a Tree

These are friends: chordal graphs.

Exactly the graphs with no chordless cycles

Lovely proof in Golumbic’s Book

Containment Graphs of Subtrees of a Tree

These are friends: comparability graphs.

Exactly the graphs with transitive orientations of the edges

Exactly all containment graphs
Filaments

What’s a filament?

Consider a line $L$ and an interval $I$ on that line
Filaments

Let $L$ be embedded in a plane $P$, and let there be another plane above and perpendicular to $P$ that intersects $P$ at exactly $L$. 
Filaments

A filament on I is a curve in the second plane connecting the endpoints of I.

If intervals are disjoint, filaments do not intersect.

If an interval contains another, then filaments may or may not intersect.
Filaments

Interval filament graphs (Gavril 2000)

Intersection graphs of filaments on intervals

Partition of non-edges - alternate definition
Interval Filament Graphs

Equivalent statements

Graph G is a:

Interval filament graph
Caterpillar overlap graph
SOG with a subtree-covering path

Recognition is NP-C
Subtree Filaments
Subtree Filament Graphs

Intersection graphs of filaments on subtrees in a tree
- If subtrees are disjoint, filaments do not intersect
- If subtrees overlap, filaments do intersect
- If one subtree contains another, filaments may or may not intersect
SFGs are SOGs

One piece of terminilogy: $T_{f_i}$
Subtree that $f_i$ is on
SFGs are SOGs

Given: $n$ Filaments $\mathcal{F} = \{f_1, f_2...f_n\}$ on tree $T = (V_T, E_T)$

Construct tree: $T'$

Construct $n$ subtrees: $\mathcal{T}' = \{t'_1, t'_2...t'_n\}$ such that:
$t'_i$ overlaps $t'_j$ iff $f_i$ intersects $f_j$
SFGs are SOGs

Construct tree: $T' = T + n$ extra nodes \{x_1...x_n\}

where

$x_i$ is adjacent in $T'$ to an endpoint of $f_i$ in $T$. 
Construct subtrees: $t'_i = T_{f_i}$

plus

$x_i$

plus

$\{x_j | T_{f_j} \subset T_{f_i} \text{ and } f_i \text{ does not intersect } f_j\}$
We must show that $t'_i$ overlaps $t'_j$ iff $f_i$ intersects $f_j$.
$t'_i$ overlaps $t'_j$ iff $f_i$ intersects $f_j$

Suppose $f_i$ intersects $f_j$
- $T_{f_i}$ intersects $T_{f_j}$
- so $t'_i$ intersects $t'_j$

But also:
- $x_i$ is in $t'_i$ but not in $t'_j$
- $x_j$ is in $t'_j$ but not in $t'_i$

Therefore $t'_i$ overlaps $t'_j$!
Suppose \( f_i \) does not intersect \( f_j \) then either \( T_{f_i} \) is disjoint from \( T_{f_j} \) or \( T_{f_j} \) is contained in \( T_{f_i} \).
$t'_i$ overlaps $t'_j$ iff $f_i$ intersects $f_j$

Say $T_{f_j}$ is contained in $T_{f_i}$

Consider the vertices in $t'_j$.
They are:

- Vertices in $T_{f_j}$
- $x_j$
- $\{x_k | T_{f_k} \subset T_{f_j} \text{ and } f_j \text{ does not intersect } f_k \}$
Say $T_{f_j}$ is contained in $T_{f_i}$.
Consider the vertices in $t'_j$.
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SFGs are SOGs

CanaDAM 2007 31/35
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- $x_j$
- $\{x_k | T_{f_k} \subset T_{f_j} \text{ and } f_j \text{ does not intersect } f_k \}$
Say $T_{f_j}$ is contained in $T_{f_i}$.
Consider the vertices in $t'_j$.
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- Vertices in $T_{f_j}$ - Are in $t'_i$
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SFGs are SOGs

CanaDAM 2007
Consider the vertices in \( t'_j \).
They are:
- Vertices in \( T_{f_j} \) - Are in \( t'_i \)
- \( x_j \) - Is in \( t'_i \)
- \( \{x_k \mid T_{f_k} \subset T_{f_j} \text{ and } f_j \text{ does not intersect } f_k \} \)
\{ x_k \mid T_{f_k} \subset T_{f_j} \text{ and } f_j \text{ does not intersect } f_k \}

Lets take a $x_k$ in $t'_j$.

$T_{f_k}$ is contained in $T_{f_j}$

$T_{f_k}$ is contained in $T_{f_i}$

then

$f_k$ does not intersect $f_i$

$x_k$ is in $t'_j$
SFG and SOGs

The two classes are equivalent - why is this surprising?

- Filaments seem more general
- Have a choice when the subtrees overlap

The proof is surprisingly straightforward

This gives us new ways to think about SOGs

*Gavril’s non-edge partitioning*
Recognition

- We suspect it’s NP-C
- Other recognition problems that are hard:
  - Interval Filament
  - OG of subtrees in tree with 3 leaves
  - OG of subtrees in tree with k leaves
  - OG of subtrees in given tree
Algorithms

- Given a subtree overlap representation we can compute:
  - Max clique/Independent Set
Exciting Future Directions!

- Complexity of Recognition
- Partition approach
- Minimal SOG triangulation and representation
- What other hard problems can we solve on SOGs?
- Do asteroidal triples come into it somehow?
- Representations of restricted size
- A general formulation for Gavril’s mixing
- Potential application to:
  - Perfect phylogeny?
  - Knowledge and social interaction?
  - Hierarchical knowledge modeling?
  - General filament to overlapping formulation
  - Mizar-Formalizing results
  - Graph algebras
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