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Series-Parallel Subgraphs of Planar Graphs

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Abstract

In this paper we show that every 3-connected (3-edge-connected) planar graph contains a 2-connected (respectively, 2-edge-connected) spanning partial 2-tree (series-parallel) graph. In contrast, a recent result by [4] implies that not all 3-connected graphs contain 2-edge-connected series-parallel spanning subgraphs.

1 Introduction

The *k-connected subgraph* problem can be described as follows: given a family $\mathcal{F}$ of graphs, a host graph G and an integer k find a k-connected spanning subgraph G' of G where $G' \in \mathcal{F}$. The k-edge-connected subgraph problem is defined similarly by replacing the vertex-connectivity of G' by edge-connectivity. The above problems are similar to the well-known minimum edge-deletion problem (see for example [5], [1] and [2]) where G' is required to have the maximum possible number of edges (with no connectivity restriction). However, unlike the latter problem little is known about the existence of solutions to the earlier problems when $\mathcal{F}$ is one of many important restricted classes of graphs. Recently, however, Dean and Seymour [4] showed the following:

Theorem 1 [DS90]. *Let $\mathcal{F}$ be a class of graphs that is closed under minors and in which each member has a chromatic number bounded by a constant $\chi_{\mathcal{F}}$. Then for all integers k and k', $k \geq k' \geq 2$, there exists a k-connected graph G that has no spanning subgraph G', $G' \in \mathcal{F}$, with minimum degree at least k'.*

Proof: Let G be the bipartite graph $(X \bigcup Y, E)$ where $|X| \geq \chi_{\mathcal{F}}(k-1)+1$, $|Y| = \binom{|X|}{k}$ and every vertex of Y is adjacent to a distinct subset of k vertices of X.

Assume that G' satisfies the conditions of the theorem and let M be a subset of $E(G')$ that covers every vertex in Y. Consider the minor H obtained by contracting M in G'. By assumption, $H \in \mathcal{F}$ and hence $\chi_H \leq \chi_{\mathcal{F}}$. That is, viewing H as a graph on the set X of vertices, H has an independent set S of size at least $\lceil |X|/\chi_{\mathcal{F}} \rceil = k$. By construction, the vertices of such a subset S of X are incident to some vertex y in Y. The degree of y in G' is at least k', $k' \geq 2$. Hence, contracting any edge incident to y introduces an edge between some two vertices of S, a contradiction. □

Hence, the k-connected and the k-edge-connected spanning subgraph problems may have no solution even if the host graph G has an arbitrarily large connectivity when $\mathcal{F}$ is any one of the following classes of graphs: graphs embeddable on an orientable surface with a bounded genus k and graphs with a bounded tree-width k (partial k-trees).

In the remaining part of this paper, we investigate both problems when the host graph is restricted to be planar. In particular, we show that the two problems have a solution when $\mathcal{F}$ is the class of series-parallel graphs, $k' = 2$ and G is any 3-connected planar graph. The results can be used to derive approximate algorithms for many NP-complete and #P-complete optimization and enumeration problems on 3-connected planar graphs. Approximating the k-terminal reliability of a probabilistic network where vertices (or edges) are subject to failure serves as a typical example. Here, one may seek a 2-connected (2-edge-connected) spanning subgraph if the vertices (respectively, edges) may fail and the remaining elements are perfectly reliable.

2 2-connected Series-Parallel Graphs

Throughout this section a graph $G = (V(G), E(G))$ is considered to be finite and loopless. A 2-terminal graph $G = (V, E)$ is a graph with two distinguished vertices, say s and t; such a graph is denoted $G(s,t)$. A graph is *series-parallel* reducible if it can be reduced to an edge by applying a sequence of series and parallel reductions and eliminations of vertices of degree 1. Series-parallel graphs are often introduced in the more general setting of *partial k-trees* (see for example [3] for definitions and some properties of partial k-trees). In this context, series-parallel graphs are precisely the class of partial 2-trees.

Given a 3-connected planar graph G we fix an embedding of G in the plane. Cycles of G are traversed in a *clockwise* direction, according to the hypothesized fixed embedding. Thus, given a cycle C of G we can refer to a path of C by listing a sequence of vertices on that path encountered when traversing C in a clockwise direction. For instance, $C_{w,x,y,z}$ is a path of C from w to z passing by x and y in that order.

Given a cycle C of G, call the graph R formed by C and the subgraph embedded in its interior a *block*. Here, we say that C *defines* R. Let R and C be as above. An *internal* path P of R is one with at least one vertex in $V(R) \setminus V(C)$, moreover, if $x \in V(P) \bigcap V(C)$ then x is an end vertex of P.

Now, suppose that s and t are two distinct terminals on $V(C)$. Our objective is to find a 2-connected spanning series-parallel subgraph $A(s,t)$ of the 2-terminal block $R(s,t)$

such that (i) $C \subseteq A(s,t)$ and (ii) $A(s,t)$ can be reduced to the edge (s,t). To this end, we introduce the following structures. Let s' and t' be two distinct vertices of C such that $C_{s',t'}$ is a subpath of either $C_{s,t}$ or $C_{t,s}$ (the two *halves* of C). Without loss of generality, we may assume that $V(C_{s',t'}) \subseteq V(C_{s,t})$. Three types of internal paths in R relative to $C_{s',t'}$ are of interest.

1. Let P be an internal path of R joining a vertex of $V(C_{s',t'}) \setminus \{s',t'\}$ to some vertex in $V(C_{s,t})$. Call the section $C_{s',t'}$ *free* (in R) with respect to $C_{s,t}$ (the half that includes $V(C_{s',t'})$) if R has no such P. By the above definition, every edge $e = (x,y)$ in C is a free path since $V(e) \setminus \{x,y\} = \phi$.

2. Let P be an internal path of R joining a vertex of $V(C_{s',t'}) \setminus \{s',t'\}$ to some vertex in $V(C_{t,s}) \setminus \{s,t\}$ (recall that $C_{t,s}$ is the half of C that does not contain $C_{s',t'}$). Such a path P is called a K_4-*completing* path of $C_{s',t'}$ in R (since, K_4 is a minor of $C \bigcup (s,t) \bigcup P$).

3. Call an internal path P which joins s' to t' *enclosing* with respect to $C_{s',t'}$ if every internal path of R that joins a vertex of $V(P) \setminus \{s',t'\}$ to some vertex of $V(P)$ is also an internal path of the block R' defined by the cycle $C_{s',t'} \bigcup P$. A simple algorithm to compute enclosing paths is presented shortly.

If R and C are as above then every possible component of the subgraph induced by $V(R) \setminus V(C)$ has a tree-like structure whose nodes correspond to blocks or vertices of R. We call such a component, together with the edges joining it to C, an *attached augmented* tree (or attached tree for short) of C in R. If T is such an attached tree and s' and t' are in $V(C_{s,t}) \bigcap V(T)$ then the following procedure finds an enclosing path P relative to $C_{s',t'}$. We first view the subgraph H induced by $E(C_{s',t'}) \bigcup E(T)$ as a separate graph with the same embedding as G. Remove from H all vertices that belong to $V(C) \setminus V(C_{s',t'})$ and then repeatedly prune all blocks and vertices of the remaining graph that are not 2-connected (with 2 vertex disjoint paths) to s' and t'. At the end, the remaining subgraph is a block of H whose cycle can be written as $C_{s',t'} \bigcup P$, where P is as required. We are now ready to prove the following

Theorem 2. *Let $R(s,t)$ be a 2-terminal block of a 3-connected planar graph G and let C be its defining cycle, where $\{s,t\} \in V(C)$. Then R has a 2-connected spanning series-parallel subgraph A that contains the cycle C and admits the (possibly new) edge (s,t).*

Proof: The proof is by induction on $|V(R) \setminus V(C)|$ the number of internal vertices in the block R. If $|V(R) \setminus V(C)| = 0$ then set $A = C$ and we are done. Now, assume the theorem holds inductively for all 2-terminal blocks (of 3-connected graphs) with fewer than k internal vertices and let R be a block with exactly k internal vertices.

Let T be an attached tree of C and let $M_{s,t}$ (also $M_{t,s}$) be the vertices of attachment of T on the section $C_{s,t}$ (respectively, $C_{t,s}$) of C. Furthermore, let M be the larger of the two sets $M_{s,t}$ and $M_{t,s}$. We may assume, without loss of generality, that $M = M_{s,t}$. Now, $|M| \geq 2$ since G is 3-connected. Let s' (respectively, t') be the closest neighbour in M to s (respectively, t) on C. Furthermore, let γ be an enclosing path in T relative to $C_{s',t'}$.

Set R_1 to be the block whose defining cycle $C_1 = C_{s',t'} \bigcup \gamma$. Similarly, set R_2 to be the block defined by the cycle $C_2 = C_{t',s'} \bigcup \gamma$ (hence, $R_1 \bigcup R_2 = R$ and $R_1 \bigcap R_2 = \gamma$). Clearly, γ is not a K_4-completing path of any section of C which is free with respect to either $C_{s,t}$ or $C_{t,s}$ since both s' and t' lie on one side of C, namely $C_{s,t}$. Moreover, every subpath of $C_{s',t'}$ which is free with respect to $C_{s,t}$ remains free on C_1 with respect to $C_{1(s',t')}$. Likewise, every subpath of $C_{t',s'}$ which is free relative to $C_{s,t}$ (or $C_{t,s}$) remains free on C_2 relative to $C_{2(s,t)}$ (respectively, $C_{2(t,s)}$).

By the induction hypothesis, the 2-terminal blocks $R_1(s',t')$ and $R_2(s,t)$ possess 2-connected spanning series-parallel subgraphs $A_1(s',t')$ and $A_2(s,t)$, respectively, such that $E(C_1) \subseteq E(A_1)$ and $E(C_2) \subseteq E(A_2)$. It is then easy to check that the combined graph $A(s,t) = A_1(s',t') \bigcup A_2(s,t)$ is a spanning 2-connected subgraph of R. In addition, γ is a free section in R_2 with respect to $C_{2(s,t)}$. Hence, R_1 can be reduced to an edge (s',t'); the reduced graph can then be further reduced to the edge (s,t) and $A(s,t)$ satisfies the theorem. $\square$

The following result is then immediate:

Corollary 1. *Let $G(s,t)$ be a 3-connected planar graph, where s and t share a common face. Then G has a 2-connected spanning series-parallel subgraph which admits the edge (s,t).*

3 2-edge-connected Series-Parallel Graphs

We now extend the above argument to show that every 3-edge-connected planar graph G contains a 2-edge-connected spanning series-parallel subgraph A. Define an *s-walk* to be a walk in which every 2-connected component is either an edge or a cycle. Figures 1.a and 1.b illustrate an s-walk and a non s-walk, respectively. A *multicycle* is a closed s-walk. Clearly, if $\mathcal{C}$ is a multicycle then every 2-connected component of $\mathcal{C}$ is a cycle. Moreover, every closed subwalk of $\mathcal{C}$ is a multicycle. Call a subgraph $\mathcal{R}$ of G formed by taking a multicycle $\mathcal{C}$ and the subgraph embedded inside it a *multiblock*. Then an *internal* walk P of $\mathcal{R}$ is one that contains at least one internal vertex of $\mathcal{R}$ and includes vertices of $\mathcal{C}$ only as end vertices.

We now introduce structures similar to the ones mentioned in the previous section. First, assume that $\mathcal{C}$ has a cycle C that includes two distinct terminals s and t. The set $E(\mathcal{C})$ can be partitioned into two subsets: $E(\mathcal{C}_{s,t})$ and $E(\mathcal{C}_{t,s})$ that are the edge sets of two graphs $\mathcal{C}_{s,t}$ and $\mathcal{C}_{t,s}$, respectively, where $V(C_{s,t}) \subseteq V(\mathcal{C}_{s,t})$, $V(C_{t,s}) \subseteq V(\mathcal{C}_{t,s})$ and $V(\mathcal{C}_{s,t}) \bigcap V(\mathcal{C}_{t,s}) = \{s,t\}$. A closed walk W of $\mathcal{C}$ lies on the boundary of both $\mathcal{C}_{s,t}$ and $\mathcal{C}_{t,s}$ if $V(C) \bigcap V(W) = \{s\}$ or $\{t\}$. We adopt the convention that any such boundary closed subwalk W of $\mathcal{C}$ belongs to $\mathcal{C}_{s,t}$. With the above convention in mind, the two walks $\mathcal{C}_{s,t}$ and $\mathcal{C}_{t,s}$ become uniquely defined; these are called the two *halves* of $\mathcal{C}$ (with respect to the 2-terminal cycle $C(s,t)$).

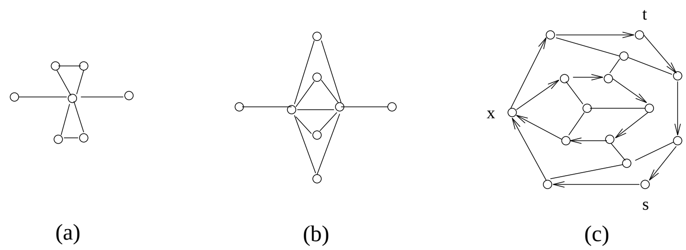

Figure 1.

Now, suppose that s' and t' are two (possibly identical) vertices on $\mathcal{C}$ and let $\mathcal{C}_{s',t'}$ be a subwalk from s' to t' having all of its edges in one half of $\mathcal{C}$. Then

1′. Call $\mathcal{C}_{s',t'}$ *free*, relative to the particular half which includes $V(\mathcal{C}_{s',t'})$, if $\mathcal{R}$ has no internal path joining a vertex in $V(\mathcal{C}_{s',t'}) \setminus \{s',t'\}$ to some vertex in that particular half.

2′. Define a K_4-*completing* path of $\mathcal{C}_{s',t'}$ to be an internal path of $\mathcal{R}$ which joins a vertex in $V(\mathcal{C}_{s',t'}) \setminus \{s',t'\}$ to some vertex, distinct from s and t, in the half of $\mathcal{C}$ that does not include $V(\mathcal{C}_{s',t'})$.

3′. Call an internal s-walk W which joins s' to t' *edge-enclosing* with respect to $\mathcal{C}_{s',t'}$ if every internal walk of R that joins a vertex of $V(W) \setminus \{s',t'\}$ to some vertex of $V(W)$ is also an internal walk of the multiblock $\mathcal{R}'$ made of the graph $\mathcal{C}_{s',t'} \bigcup W$ and its interior edges.

Again, given a 2-terminal multiblock $\mathcal{R}(s,t)$ our objective is to find a 2-edge-connected spanning series-parallel subgraph $A(s,t)$ where $\mathcal{C} \subseteq A(s,t)$ and $A(s,t)$ admits the edge (s,t). Such a subgraph $A(s,t)$ may not exist, however, for any arbitrarily chosen multicycle that defines $\mathcal{R}$. The graph $\mathcal{R}$ illustrated in Figure 1.c is an example, given that we choose $\mathcal{C}$ to be the multicycle marked by arrows. Therefore, we introduce the following type of restricted multicycles. We say that $\mathcal{C}$ is a *valid* multicycle for $\mathcal{R}(s,t)$ if $\mathcal{C}$ defines $\mathcal{R}(s,t)$ and whenever C_1 and C_2 are two cycles of $\mathcal{C}$ with $E(C_2)$ lying in the interior of C_1 then the interior of C_2 is empty.

Clearly, if $\mathcal{C}$ is valid for $\mathcal{R}$ and $\mathcal{C}'$ is a closed subwalk of $\mathcal{C}$ that defines a subgraph $\mathcal{R}'$ then $\mathcal{C}'$ is valid for $\mathcal{R}'$. In addition, any cycle that defines $\mathcal{R}$ is a valid multicycle. We are now ready to proof the following theorem.

Theorem 3. *Let $\mathcal{R}(s,t)$ be a 2-terminal multiblock of a 3-edge-connected planar graph G. Let $\mathcal{C}$ be a minimal valid multicycle for $\mathcal{R}(s,t)$. Then $\mathcal{R}$ has a 2-edge-connected spanning series-parallel graph $A(s,t)$ that contains $\mathcal{C}$ and admits the (possibly new) edge (s,t).*

Proof: We induct on $|V(\mathcal{R}) \setminus V(\mathcal{C})|$. The basis and the induction hypothesis are simple to deduce. For the induction step we consider the following cases.

First, suppose that $\mathcal{C}$ has a cutvertex x. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two multicycles of $\mathcal{C}$ such that $E(\mathcal{C}) = E(\mathcal{C}_1) \bigcup E(\mathcal{C}_2)$ and $V(\mathcal{C}_1) \bigcap V(\mathcal{C}_2) = \{x\}$. Similarly, let $\mathcal{R}_1$ and $\mathcal{R}_2$ be the two multiblocks defined by $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. Clearly, $\mathcal{C}_1$ ($\mathcal{C}_2$) is valid for $\mathcal{R}_1$ (respectively, $\mathcal{R}_2$). If s and t belong to one of the two multiblocks, say $\mathcal{R}_1$, then apply the induction hypothesis to $\mathcal{R}_1(s,t)$ and $\mathcal{R}_2(x,y)$ where y is any vertex of $V(\mathcal{C}_2) \setminus \{x\}$. Otherwise, we may assume, without loss of generality, that $s \in \mathcal{R}_1$ and $t \in \mathcal{R}_2$. We apply the induction to $\mathcal{R}_1(s,x)$ and $\mathcal{R}_2(t,x)$. In either case, the theorem follows easily.

Otherwise, $\mathcal{C}$ does not have a cutvertex. Then $\mathcal{C}$ has a distinguished cycle C which defines $\mathcal{R}$. Similar to the proof of theorem 2, let T be an attached tree of $\mathcal{C}$ and let $M_{s,t}$ (also $M_{t,s}$) be the edges of attachment of T on $\mathcal{C}_{s,t}$ (respectively, $\mathcal{C}_{t,s}$). Set M to be the larger of the two sets (hence, $|M| \geq 2$). In addition, let $V_{MC} = V(M) \bigcap V(C)$ and $\mathcal{C}_{MC}$ to be the half of $\mathcal{C}$ which includes V_{MC}.

Now, V_{MC} has two vertices s' and t' ($s' = t'$ is a possibility) such that $\mathcal{C}_{s',t'}$ is a subwalk (from s' to t') of C_{MC} and $V_{MC} \subseteq V(\mathcal{C}_{s',t'})$. In addition, let γ be a *minimal* edge-enclosing s-walk of $\mathcal{C}_{s',t'}$ in T (if $s' = t'$ then γ is a multicycle). The minimality condition ensures that γ has no subwalk γ' such that $\mathcal{C}_{s',t'} \bigcup \gamma'$ encloses edges in $E(\gamma) \setminus E(\gamma')$.

If $s' \neq t'$ then set $\mathcal{R}_1(s',t')$ to be the multiblock defined by $\mathcal{C}_1 = \mathcal{C}_{s',t'} \bigcup \gamma$. Otherwise, ($s' = t'$) choose r' to be a vertex of γ distinct from s' and set $\mathcal{R}_1(s',r')$ to be the multiblock defined by $\mathcal{C}_1 = \gamma$. Also, set $\mathcal{C}_2$ such that $E(\mathcal{C}_2) = E(\gamma) \bigcup (E(\mathcal{C}) \setminus E(\mathcal{C}_{s',t'}))$ and let $\mathcal{R}_2(s,t)$ be the multiblock defined by $\mathcal{C}_2$ minus the vertices and edges which appear in the interior of $\mathcal{R}_1$. One may then verify that

1. $\mathcal{C}_1$ and $\mathcal{C}_2$ are minimal valid multicycles for $\mathcal{R}_1(s',t')$ (or $\mathcal{R}_1(s',r')$) and $\mathcal{R}_2(s,t)$, respectively,

2. γ is not a K_4-completing path of any free section (relative to $\mathcal{C}_{s,t}$ or $\mathcal{C}_{t,s}$) of $\mathcal{C}$,

3. every subwalk of $\mathcal{C}_{s',t'}$ which is free with respect to $\mathcal{C}_{s,t}$ remains free on $\mathcal{C}_1$ with respect to $\mathcal{C}_{1(s',t')}$ (or $\mathcal{C}_{1(s',r')}$),

4. every subwalk of $E(\mathcal{C}) \setminus E(\mathcal{C}_{s',t'})$ which is free relative to $\mathcal{C}_{s,t}$ (or $\mathcal{C}_{t,s}$) remains free on $\mathcal{C}_2$ relative to $\mathcal{C}_{2(s,t)}$ (respectively, $\mathcal{C}_{2(t,s)}$) and

5. the graph $\mathcal{C}'_2$ on the set $E(\gamma) \bigcup (E(\mathcal{C}_{MC}) \setminus E(\mathcal{C}_{s',t'}))$ of edges is a half of $\mathcal{C}_2$. Moreover, γ is free in $\mathcal{R}_2$ with respect to $\mathcal{C}'_2$.

Hence, applying the induction hypothesis results in two series-parallel graphs $A_1(s',t')$ (or $A_1(s',r')$ if $s'=t'$) and $A_2(s,t)$ whose union satisfies the theorem. □

We then have

Corollary 2. *Let $G(s,t)$ be a 3-edge-connected planar graph, where s and t share a common face in some embedding of G. Then G has a 2-edge-connected spanning series-parallel subgraph which admits the edge (s,t).*

Proof: Fix an embedding of $G(s,t)$ where s and t lie on the exterior face f. Choose $\mathcal{C}$ to be the multicycle whose elements lie on f and apply Theorem 3.

4 Concluding Remarks

A direct extension to the problems considered above can be stated as follows:

Conjecture 1. *Every 4-connected planar graph contains a 3-edge-connected spanning partial 3-tree.*

At present, the above conjecture seems to be open. To investigate the combinatorics of the problem further, we present the following equivalent (but more restricted) version of the Conjecture.

Conjecture $1'$. *Every 4-connected planar graph G with 3 vertices v_1, v_2 and v_3 sharing a face f, $|V(f)| \geq 4$, contains a 3-edge-connected partial 3-tree that admits the triangle (v_1,v_2,v_3).*

Lemma. *Conjectures* 1 *and* $1'$ *are equivalent.*

Proof: The proof uses the following structural properties of the edge-labelled octahedron, denoted P_6, illustrated in Figure 2:

i. every 3-edge-connected spanning subgraph H of P_6 contains a *target* vertex t incident with 3 (or more) edges whose nearest ends are labelled 1, 2 and 3 and

ii. H has a minor isomorphic to K_4 having the set $\{t\}$ as one of its vertices.

Let G, f, v_1, v_2 and v_3 be as in Conjecture $1'$. Fix an embedding of G in the plane with f as an exterior face. Let v_4 be some vertex in $V(f) \setminus \{v_1,v_2,v_3\}$. Consider the new graph G' obtained by replacing each node in P_6 by a copy of G such that for every i, $1 \leq i \leq 4$, vertex v_i in each copy is incident to the edge whose nearest end is labelled i. It is then easy to verify that G' is 4-connected.

We now show that if G' has a 3-edge-connected spanning partial 3-tree (as in Conjecture 1) then G has a 3-edge-connected partial 3-tree that admits the triangle (v_1,v_2,v_3) (as in Conjecture $1'$). Assume that Conjecture 1 is true and let H' be a 3-edge-connected spanning partial 3-tree of G'. By property (i), H' contains a subgraph H of some target copy of G which is attached to the rest of H' by the vertices v_1, v_2 and v_3. Moreover, property (ii) ensures that H' contains a minor $H_c = (V(H), E(H) \bigcup \{(v_1, v_2), (v_1, v_3), (v_2, v_3)\})$. Now, H_c is a 3-edge-connected partial 3-tree (since partial 3-trees are closed under minors). Hence, H_c can be reduced to the triangle (v_1, v_2, v_3) and Conjecture $1'$ holds true. □

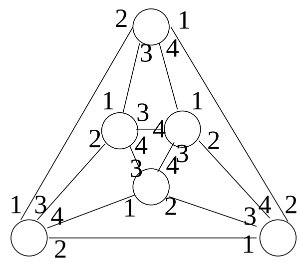

Figure 2.

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