CMPUT 403: Unweighted Graph Algorithms

Zachary Friggstad and Marlos C. Machado

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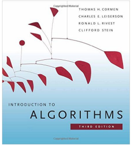
References

Chapter 4: Graph (Section 4.2)

Chapter 22: Elementary Graph Algorithms



Unweighted Graphs

Note

Many code snippets here use `C++ 11` features. Compile with the flag `-std=c++11` if using `g++`.

Throughout, $n = \#$ vertices, $m = \#$ edges.

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Adjacency List Representation of a Graph

```
//without c++11 you may need to add a space between >>
typedef vector<vector<int>> graph;
...
graph g(n);          //create a graph with n vertices
g[u].push_back(v);   //add v as a neighbour of u
```

For undirected graphs, just add both directions of an edge (u, v).
Requires $\Theta(n + m)$ space.

Depth-First Search

Find all vertices reachable from vertex v.

```
//the vertices that are reached in the search
vector<bool> reached(n, false);
graph g;

void dfs(int u) {
   if (!reached[u]) {
      reached[u] = true;
      for (auto w : g[u]) dfs(w);
   }
}
...
dfs(v);
```

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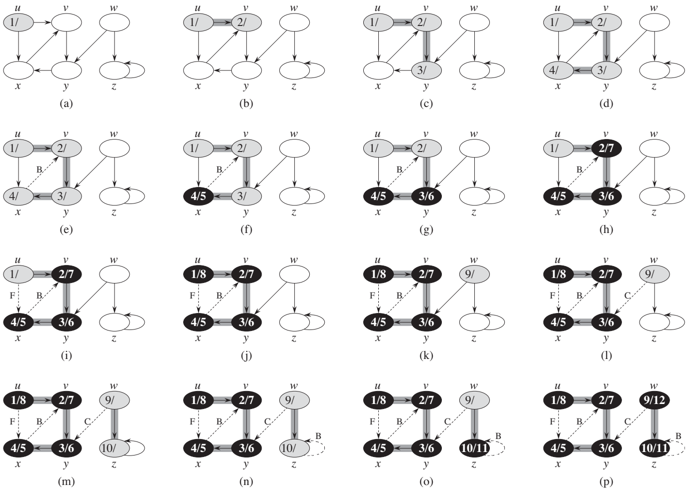
```

If we record the vertex that discovered u, we can reconstruct paths.

Runs in $O(n + m)$ time.

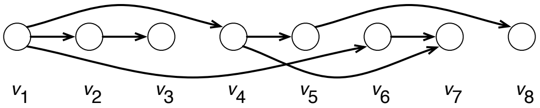
Depth-First Search



Example from CLRS (page 542, Figure 22.4)

Applications of DFS: Topological Sorting

Order the vertices so all edges point left-to-right.



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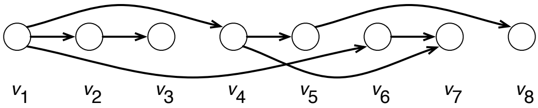

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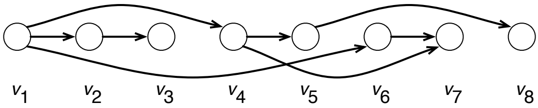

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- Begin a DFS. Just before returning from a recursive call (i.e. just after the `for` loop) `push_back` the vertex u to the end of a vector.

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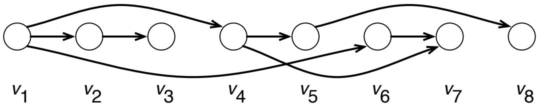


Impossible to do if there is a cycle. Otherwise, the following works.

- Begin a DFS. Just before returning from a recursive call (i.e. just after the `for` loop) `push_back` the vertex u to the end of a vector.
- Repeat, starting with an unvisited vertex each time, until all vertices are visited.

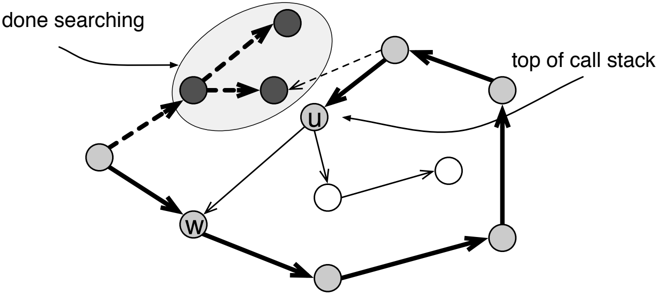
```
vector<int> order; //initially empty

void topo_sort(int u) {
   if (!reached[u]) {
      reached[u] = true;
      for (auto w : g[u]) topo_sort(w);
      order.push_back(u);
   }
}

...

for (int u = 0; u < n; u++)
   if (!reached[u])
      topo_sort(u);
reverse(order.begin(), order.end()); //#include <algorithm>
```

If u is ordered after w for some edge (u, w), it must be that the recursive call with w was on the call stack when u was being processed. (**Why?**)



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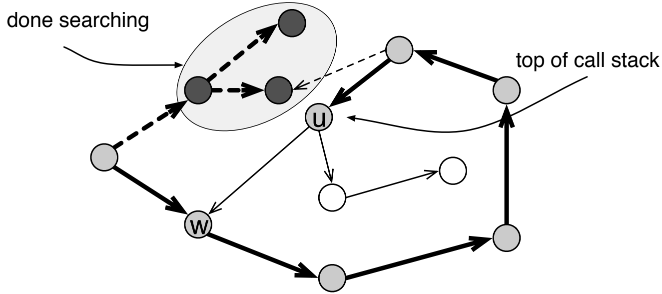

If w is on the call stack when u is being processed, there is a path from w to u. Completing this path with the edge (u, w) yields a cycle.

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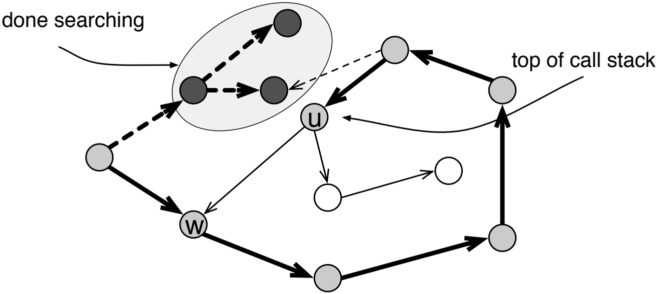


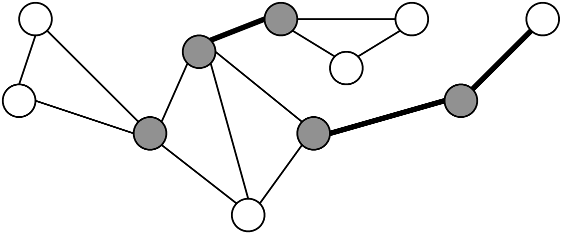
If w is on the call stack when u is being processed, there is a path from w to u. Completing this path with the edge (u, w) yields a cycle.

Thus

If the graph has no cycles, this will topologically sort all vertices.

Articulation Points & Bridges

An articulation point in an undirected, connected graph is a vertex whose removal leaves a disconnected graph.



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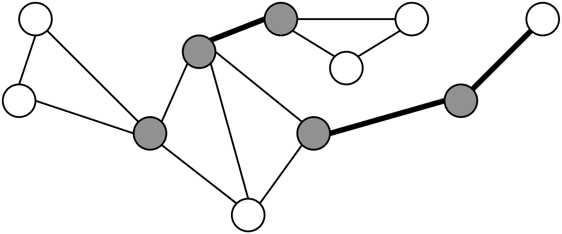
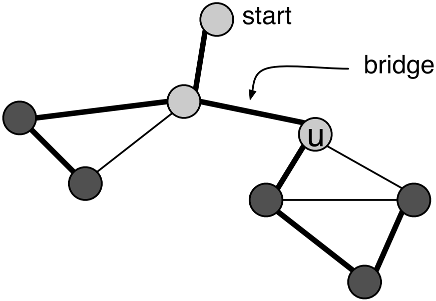


A bridge is an edge whose removal leaves a disconnected graph.

Can find all bridges and articulation points in $O(n + m)$ time via DFS.

A bridge will always be a tree edge in a DFS (actually, in any spanning tree).



Picture: no edge of a descendent of u in the search reached a non-descendent. So the parent edge of u is a bridge.

Run a DFS, record the order the vertices were discovered.

Return the earliest discovery time of any vertex adjacent to a descendant of u. This indicates if some descendant is adjacent to a non-descendant.

```
vector<int> found(n, -1); // discovery time
int cnt = 0;

int bridges(int u, int p) {
   if (found[u] != -1) return found[u];
   int mn = found[u] = cnt++; //record u's discovery time
   for (auto w : g[u])
     mn = min(mn, bridges(w, u));
   if (mn == found[u] && p != -2)
      // (p, u) is a bridge, process it how you want
   return mn;
}
...
bridges(0, -2); //start the search from any vertex
```

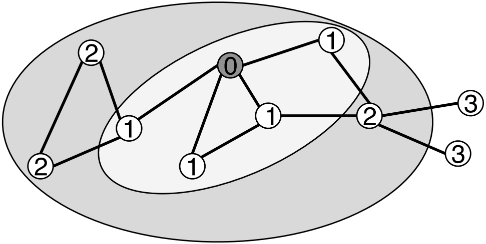
Other DFS Applications

- Find all articulation points in a graph (good exercise).
- Find the *strongly connected components* of a directed graph.
- Compute pre/post order traversals of a tree.
- Simple code for augmenting a bipartite matching (later lecture).

All of these can be implemented to run in $O(n + m)$ time.

Breadth-First Search

A *breadth-first search* will explore the vertices in increasing order of their shortest path distance from the start vertex.



- Load up the start vertex in a queue q.
- While q is not empty, extract the front vertex and add all of its unvisited neighbours to the back of q.

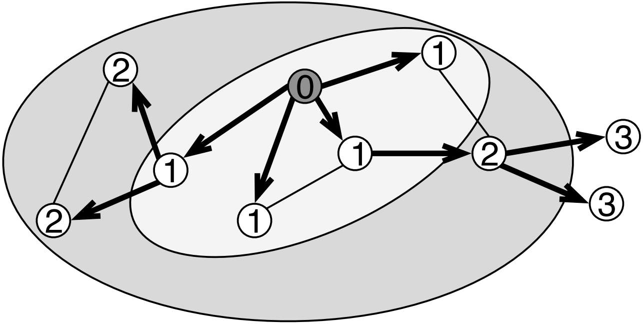
```
queue<int> q; //#include <queue>
vector<int> prev(n, -1);

q.push(v);    //v is the start vertex in the search
prev[v] = -2; //signals "root of search"

while (!q.empty()) {
   int curr = q.front();
   q.pop();
   for (auto succ : g[curr])
      if (prev[succ] == -1) {
         prev[succ] = curr;
         q.push(succ);
      }
}
```

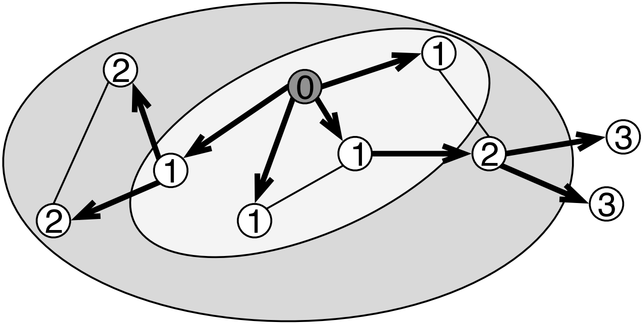
Now $prev[u]$ for $u \neq v$ is the vertex prior to u on a shortest $v - u$ path.

Also runs in $O(n + m)$ time.



A thick arrow from u to w indicates $prev[w] = u$.

The unique path using thick arrows from the start vertex (**dark**) to any vertex is a shortest path in the graph.



A thick arrow from u to w indicates $prev[w] = u$.

The unique path using thick arrows from the start vertex (**dark**) to any vertex is a shortest path in the graph.

Though we illustrated with an undirected graph, the same algorithm also finds shortest paths in directed graphs.

To Come...

Next week

Algorithms for weighted graphs.

- Dijkstra's algorithm for shortest paths.
- Floyd-Warshall for all-pairs shortest paths.
- Bellmand-Ford: handling negative weight cycles.
- Minimum Spanning Trees: Kruskal's Algorithm

Later in the course

- Bipartite matching: unweighted and weighted.
- Network flow: max-flow/min-cut.