CMPUT 403: Geometry

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Points

```
typedef complex<double> point; //#include <complex>
point p(1.0, 5.7);
p.real(); // x component
p.imag(); // y component
```

Helpful operations

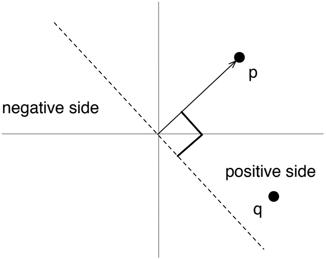
```
point p, q;
p+q; // exactly what you expect
p *= polar(1, theta); //rotate p by theta radians
abs(p-q); //distance between p and q
arg(p); //angle from positive x-axis, measured in (-pi, pi]
```

One Annoyance: we can't update the x and y components without creating a new point.

Dot Product

$p \circ q = p_x \cdot q_x + p_y \cdot q_y = ||p|| \cdot ||q|| \cdot \cos\theta$

```
double dot(point p, point q) { return real(p*conj(q)); }
```

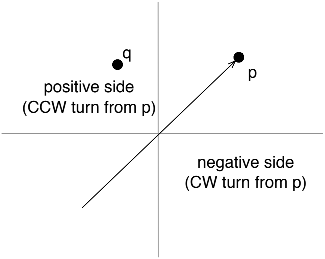


- $p \circ q = 0$ if $p \perp q$ (or one of them is 0)
- $p \circ p = ||p||^2$
- $p \circ q > c$: q lies in the *same* general direction as p
- $p \circ q < 0$: q lies in the *opposite* general direction as p

Cross Product

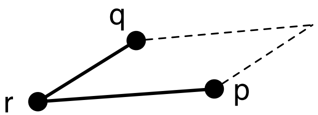
$$p \times q = p_x \cdot q_y - p_y \cdot q_x = ||p|| \cdot ||q|| \cdot \sin\theta$$

```
double cross(point p, point q) { return imag(p*conj(q)); }
```

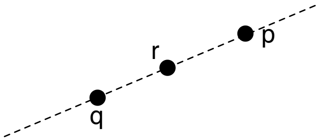


- $p \times q = 0$ if p and q are collinear with 0 (or one of them is 0)
- $p \times q > 0$: $\angle(p, q)$ is *counterclockwise*
- $p \times q < 0$: $\angle(p, q)$ is *clockwise*

$\frac{1}{2}|(p-r)\times(q-r)|$ is the area of the triangle with corners p, q, r.



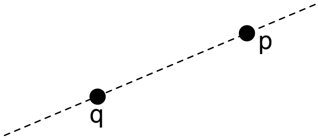
Tip: p lies on the line passing through points q and r **if and only if** $(q-p)\times(r-p)=0$.



Additionally: if p lies on this line then it lies between q and r **if and only** if $(q-p)\circ(r-p)\leq 0$.

Parametric representation of a line passing through p and q:

$$r(t) = (1-t) \cdot p + t \cdot q, \quad t \in \mathbb{R}$$



Note $r(0) = p$ and $r(1) = q$.

The line segment connecting p and q is parameterized this way, with the restriction $0 \leq t \leq 1$.

Given two lines passing through p, q and p', q' respectively, compute their intersection point (if any).

Solve $(1-t)\cdot p + t\cdot q = (1-t')p' + t'\cdot q'$ for $t, t' \in \mathbb{R}$.

$$\begin{bmatrix} q_x - p_x & p'_x - q'_x \\ q_y - p_y & p'_y - q'_y \end{bmatrix} \cdot \begin{bmatrix} t \\ t' \end{bmatrix} = \begin{bmatrix} p'_x - p_x \\ p'_y - p_y \end{bmatrix}$$

Use Cramer's rule: for $A \cdot x = b$ we have

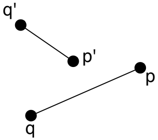
$$x_i = \frac{\det A_i}{\det A}$$

Where A_i is the matrix obtained by replacing column i of A with b.

Tip: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$.

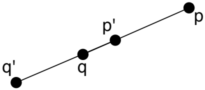
$\det A = 0$ means the lines are parallel.

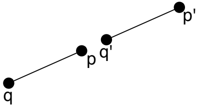
Computing the intersection of two line segments.



- If the infinite lines intersect (i.e. not parallel), get values t, t'.
- Ensure $0 \leq t \leq 1$ and $0 \leq t' \leq 1$.

If lines are colinear:

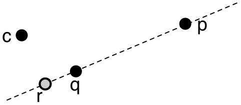




- If $p = p'$ and $q = q'$ or vice versa, they overlap.
- Else if a point is in the interior of the other segment, they overlap.
- Else if the segments share a point, they touch only at that point.
- Else they do not touch.

Closest point on a line.

Given a line passing through p, q and another point c, find the point r on the line $\overline{pq}$ nearest to c.



Minimize $||(1-t)p + t \cdot q - c||^2$ over $t \in \mathbb{R}$ (avoids the square root).

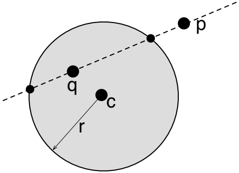
Expanding, this is just minimizing some quadratic:

$$a \cdot t^2 + b \cdot t + c$$

Tip: the minimum is at $-\frac{b}{2a}$. If $a = 0$, then $p = q$.

Line-Circle Intersection

A line $\overline{pq}$ and a circle (r, c) (radius/center point) does the line puncture the circle?



Solve $||(1-t) \cdot p + t \cdot q - c||^2 = r^2$ for t.

Rearrange: $a \cdot t^2 + b \cdot t + c = 0$. $a = 0$ means $p = q$.

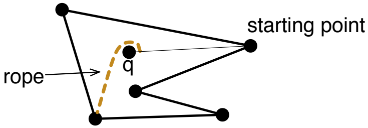
Tip: Use the quadratic formula

$$\frac{-b \pm \sqrt{\Delta}}{2a} \quad \text{where} \quad \Delta = b^2 - 4ac.$$

$\Delta < 0$: line misses, $\Delta = 0$: line tangent, $\Delta > 0$: line punctures.

Point q in Polygon $P : p_1, \ldots, p_n$?

Intuition: walk about the polygon with a rope taut with “post” q.

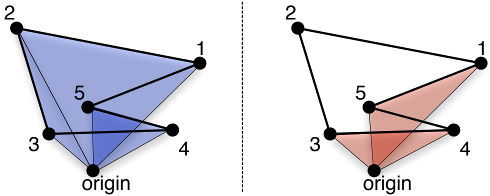


If q is inside, the rope wraps around once. If q is outside, the rope doesn't wrap around.

```
double delta = 0;
for (int i = 0, j = n-1; i < n; j = i++)
   delta += arg((p[i]-q) / (p[j]-q)); //change in angle
return fabs(delta) > 1; //|delta| be 0 or 2*pi
```

Have to add a bit more to check if q lies on the boundary.

Area of a Polygon $P : p_1, \ldots, p_n$



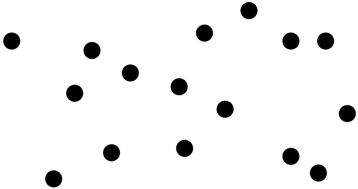
blue triangle: count the area **positively** (CCW turn about origin)
red triangle: count the area **negatively** (CW turn about origin)

```
double area = 0;
for (int i = 0, j = n-1; i < n; j = i++)
   //signed area of triangle [origin, p[i-1], p[i]]
   area += cross(p[i], p[j])*0.5;
return fabs(area);
```

Points outside are "cancelled", points inside are counted +1 times (after cancelling). The net sum is the area of P.

Convex Hull

Given points $p_1, \ldots, p_n$, what is their *convex hull*?



Output: The points on the convex hull in CCW order.

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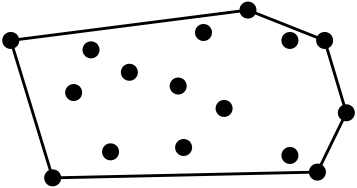
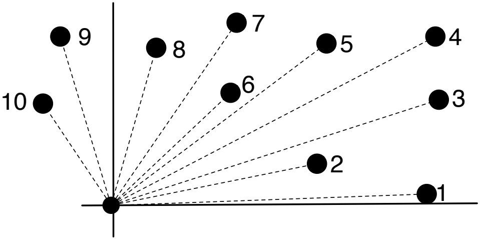


Output: The points on the convex hull in CCW order.

Convex Hull

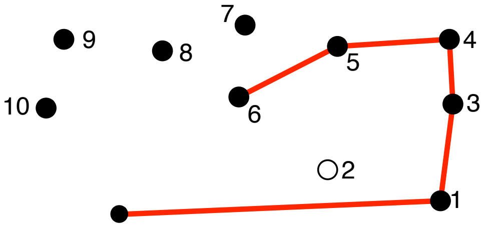
The bottom-most point must be on the convex hull. If there are many, choose the leftmost. Suppose this point is $(0, 0)$ by shifting all points if needed.



Sort all other points by their angle with the positive x-axis. To check if $p_i < p_j$ according to this order, just check $p_i \times p_j > 0$.

The origin and the first point after sorting are on the hull for sure.

Build up the hull one point at a time: add the points in the sorted order.



If one creates a clockwise turn with the previous two on the hull, then pop the end of the current hull.

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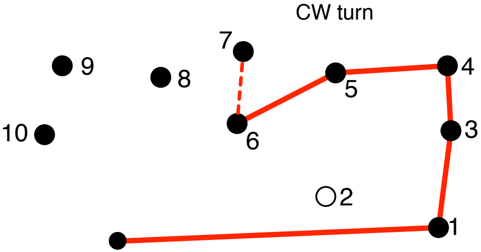

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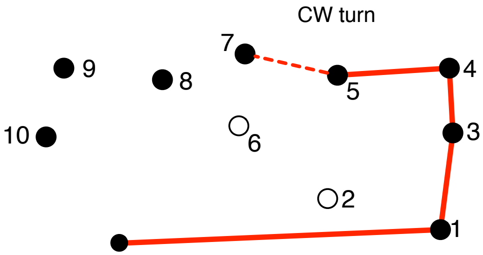

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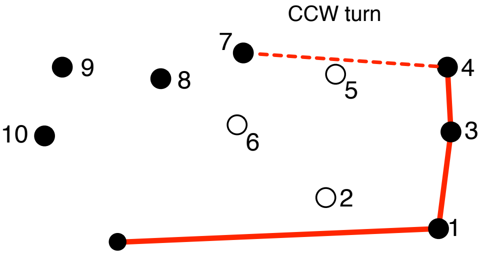

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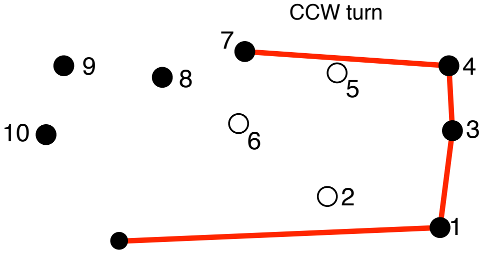

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```
bool cmp(point a, point b) { return cross(a, b) > 0; }

point p[MAXN], hull[MAXN];
//Assume p[n-1] is the lowest point (breaking ties by
//taking leftmost) and no 3 points are collinear

for (int i = 0; i < n-1; ++i) p[i] -= p[n-1];
sort(p, p+n-1, cmp); //sort by angle from x-axis

hull[0] = point(0,0);
hull[1] = p[0];
int hs = 2;

for (int i = 1; i < n-1; ++i) {
  while (cross(hull[hs-1]-hull[hs-2],p[i]-hull[hs-2]) < 0)
     --hs;
  hull[hs++] = p[i];
}
for (int i = 0; i < hs; ++i) hull[i] += p[n-1];
```

Running time: $O(n \log n)$

Tips

Never use `float`, not enough precision. Use `double`.

Sometimes values that should be equal aren't quite due to floating point errors.

Use the following for safer comparisons.

```
#define EPS 1e-8
double a, b;
if (fabs(a-b) < EPS) ...   //check if a == b
if (a + EPS < b) ...       //check if a < b
if (a < b + EPS) ...       //check if a <= b
```

Can there be 3 collinear points? Can points be equal? How can you deal with this?

Missing Topics

Voronoi diagrams/Delaunay triangulations, 3D convex hull, closest pair of points, furthest pair of points, triangulating a polygon.

Next Time

Number Theory

To Come

- Combinatorics and Arithmetic
- String Processing
- Matchings and Network Flow

Possible Misc. Topics

Probability, counting MSTs, edge colourings, voronoi diagrams, simplex for linear programming, Nim + extensions, matroids.