CMPUT 675: Approximation Algorithms **Fall 2011**

Lecture 12-13, (Oct. 18 and 20, 2011): MST, Min-cost Bounded Degree ST

Lecturer: Mohammad R. Salavatipour *Scribe: Xida Chen*

12.1 Minimum Spanning Tree

Recall the following LP relaxation for the Minimum Spanning Tree (MST) problem from last lecture:

$$\begin{array}{lrcll} \min & \sum_{e\in E} c_e \cdot x_e & & & \\ \text{s.t.} & x(E(v)) & = & |V|-1 & \\ & x(E(S)) & \leq & |S|-1 & \forall S\subset V, |S|\geq 2 \\ & x_e & \leq & 1 & \\ & x_e & \geq & 0 & \end{array}$$

We call this LP_{MST}. Our goal is to prove that:

Theorem 1 *LP_{MST} is integral, i.e. every bfs of this LP is integral.*

We know that any bfs is uniquely determined by n linearly independent tight constraints (where n is number of variables of the LP). Since we have exponentially many constraints in this LP, a bfs may be satisfying many of them with equality (i.e. being tight). We want a "good" set of linearly independent tight constraints defining it. The notion of "good" here will be clear soon. First, observe that in any bfs, we can safely delete any edge $e \in E$ with $x_e = 0$ from the graph. So we can assume that every edge of G has $x_e > 0$. Our goal is to show that there are at most $|v|-1$ linearly independent tight constraints which implies that there are at most $|v|-1$ non-zero variables. Since for any set S with size $|S| = 2$, the condition $x(E(S)) \leq |S|-1$ implies the value of that edge must be at most 1 and because $x(E(v)) = |v|-1$ we get that all the $|v|-1$ non-zero variables must have value exactly 1, i.e. the bfs is integral. Before we present the algorithm for this LP, we introduce some Lemmas first. Recall that:

Definition 1 *Two sets X, Y over a ground set U are called crossing if $X \cap Y \neq \emptyset$, $X - Y \neq \emptyset$, and $Y - X \neq \emptyset$. A family of sets is called laminar if no two sets in the family cross.*

Lemma 1 *Suppose U is a ground set and $|U| = n$ and $\mathcal{L}$ is a laminar family of subset of U without subsets of size 1, then $|\mathcal{L}| \leq n-1$ ($n = |E|$)*

Proof. We proof this lemma by induction. First we define $S \in \mathcal{L}$ to be maximal if $\nexists S' \in \mathcal{L}$ *s.t.* $S \subset S'$. Let $S_1, ..., S_m$ be collection of maximal sets of $\mathcal{L}$ (i.e. roots of the tree in the corresponding forest). We can see that each S_i contains at most $|S_i|-1$ sets of $\mathcal{L}$ since the difference of each set and its parrent is at least one element and leaf nodes contain at least two elements. Hence, we have:

$$|\mathcal{L}| \leq \sum_{i=1}^{m}(|S_i - 1|) \leq \sum_{i=1}^{m}|S_i| - m \leq n-1$$

■

The following observation can be proved by noting that every type of edge contributes the same amount to each side of the equation below:

Observation 1 *For any two sets of vertices $X, Y \subseteq V$. Then*

$$x(E(X)) + x(E(Y)) = x(E(X \cup Y)) + x(E(X \cap Y)) - x(\delta(X, Y))$$

where $\delta(X, Y)$ denotes the edges between X and Y.

An immediate corollary is:

Corollary 1 *For any two sets $X, Y \subseteq V$: $x(E(X)) + x(E(Y)) \leq x(E(X \cup Y)) + x(E(X \cap Y))$.*

Let x be a bfs of the LP_{MST} with $x_e > 0$ for all edges $e \in E$. Let $\mathcal{F} = \{S | x(E(S)) = |S| - 1\}$ be the family of all tight constraints of the LP. For each set S we use $\chi(E(S))$ to denote the characteristic vector of $E(S)$ of size $|E|$:

$$\chi(E(S)) = \begin{cases} 1 & if\ e \in E(S) \\ 0 & o.w. \end{cases}$$

Lemma 2 *If $S, T \in \mathcal{F}$ and $S \bigcap T \neq \emptyset$, then both $S \bigcup T$ and $S \bigcap T$ are in $\mathcal{F}$, furthermore*

$$\chi(E(S)) + \chi(E(T)) = \chi(E(S \bigcup T)) + \chi(E(S \bigcap T)).$$

Proof.

$$\begin{aligned} |S| - 1 + |T| - 1 &= x(E(S)) + x(E(T)) \\ &\leq x(E(S \bigcup T)) + x(E(S \bigcap T)) \\ &\leq |S \bigcup T| - 1 + |S \bigcap T| - 1 \\ &= |S| - 1 + |T| - 1 \end{aligned}$$

Here the first inequality is inferred by the observation stated above, and the second inequality is inferred by LP.

From the above equations we can see that the equality holds everywhere. Hence, we have $x(E(S \bigcup T)) = |S \bigcup T| - 1$ and $x(E(S \bigcap T) = |S \bigcap T| - 1$. It implies that both $S \bigcup T$ and $S \bigcup T$ are tight sets and are in $\mathcal{F}$.

From the observation we have $x(\delta(S, T)) = 0$, which means that $\nexists e = (u, v) \in E$, s.t. $u \in S$ and $v \in T$. Therefore, we conclude that

$$\chi(E(S)) + \chi(E(T)) = \chi(E(S \bigcup T)) + \chi(E(S \bigcap T)).$$

■

We use $span(\mathcal{F})$ to denote the vector space of those sets $S \in \mathcal{F}$ i.e. the vector space of $\{\chi(E(S)) : S \in \mathcal{F}\}$.

Lemma 3 *If $\mathcal{L}$ is a maximal laminar sub-family of $\mathcal{F}$, then $span(\mathcal{L}) = span(\mathcal{F})$*

Proof. We proof this lemma by contradiction. Suppose $\mathcal{L}$ is a maximal laminar sub-family, but $span(\mathcal{L}) \subset span(\mathcal{F})$. For each $S \notin \mathcal{L}$, we define

$$intersect(S, \mathcal{L}) = |\{T \in \mathcal{L} | T\ intersect\ S\}|.$$

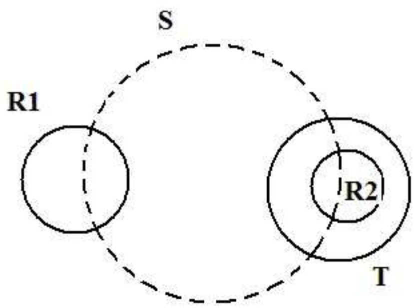

Figure 12.1: Cases for proof of Proposition 1

There must be exists set $S \in \mathcal{F}$ with characteristic vector $\chi(E(S)) \notin span(\mathcal{L})$. Pick such set S with smallest $intersect(S, \mathcal{L})$. We will have $intersect(S, \mathcal{L}) \geq 1$ since $\mathcal{L}$ is maximal. Let T be any set of $\mathcal{L}$ intersecting S, using Lemma 2 both $S \bigcup T$ and $S \bigcap T$ are tight sets. We will prove the following Proposition shortly.

Proposition 1

$$intersection(S \cap T,\ \mathcal{L}) < intersection(S,\ \mathcal{L})$$

$$intersection(S \cup T,\ \mathcal{L}) < intersection(S,\ \mathcal{L})$$

For now assume this proposition is true. Applying Lemma 2 to S and T, we get both $S \cap T$ and $S \cup T$ are in $\mathcal{F}$. So using this proposition and by minimality of $intersect(S, \mathcal{L})$, both $S \cap T$ and $S \cup T$ are in $span(\mathcal{L})$. On the other hand, $\chi(E(S)) + \chi(E(T)) = \chi(E(S \cap T)) + \chi(E(S \cup T))$. Since $\chi(E(S \cap T))$ and $\chi(E(S \cup T))$ are in $span(\mathcal{L})$ and $T \in \mathcal{L}$, we must have $\chi(E(S)) \in span(\mathcal{L})$, a contradiction.

So it only remains to prove the above proposition. Consider the case shown in Figure 12.1, we can see that for every $R \in \mathcal{L}$ and $R \neq T$, anything intersecting $S \bigcap T$ and $S \bigcup T$ must be intersecting S. Therefore, the contribution to $intersect(S \bigcap T, \mathcal{L})$ is no larger than that of $intersect(S, \mathcal{L})$. Also, the contribution to $intersect(S \bigcup T, \mathcal{L})$ is no larger than that of $intersect(S, \mathcal{L})$. However, S intersect T, but $S \bigcup T$ doesn't, hence $intersect(S \bigcup T, \mathcal{L}) < intersect(S, \mathcal{L})$. The same argument holds for $S \bigcap T$, and hence $intersect(S \bigcap T, \mathcal{L}) < intersect(S, \mathcal{L})$

■

Thus, we obtain the following:

Lemma 4 *Let x be a bfs of the LP_{MST} with $x_e > 0$ for all $e \in E$ and let $\mathcal{F} = \{S | x(E(S)) = |S| - 1\}$. Then there is a laminar family $\mathcal{L} \subseteq \mathcal{F}$ such that:*

1. *vectors of $\{\chi(E(S)) | S \in \mathcal{L}\}$ are linearly independent, and*
2. *$span(\mathcal{L}) = span(\mathcal{F})$*
3. *$|\mathcal{L}| = |E|$*

12.1.1 Iterative Algorithm

Here we describe an iterative algorithm to obtain a tree T from a bfs of the LP_{MST}; this is done by picking edges with value 1 in the LP iteratively:

Iterative Algorithm I

$F \leftarrow \emptyset$
while $V(G) \neq \emptyset$ **do**
  Find a bfs x of LP_{MST} and remove any edge e with $x_e = 0$
  Find a vertex v with degree 1, say $e = uv$; then $G \leftarrow G - \{v\}$ and $F \leftarrow F \cup \{e\}$

Figure 12.2: First Iterative Algorithm for Minimum Spanning Tree

Lemma 5 *For any basic feasible solution x from step 3 with all $x_e > 0$, there is a vertex v with $deg(v) = 1$. (It implies that every iteration there exist a v in step 4).*

Proof. We proof this lemma by contradiction. Suppose $\forall v \in V, deg(v) \geq 2$. Then $|E| = \frac{1}{2}\sum_{v \in V} deg(v) \geq |V|$. Since there is no edge with $x_e = 0$, each tight constraint $x(E(S)) = |S| - 1$. For each basic feasible solution, there is a laminar family $\mathcal{L}$ with $|\mathcal{L}| = |E| \geq |V|$. Note that $\mathcal{L}$ cannot have sets of size 1 because every constraint for laminar family is not for singletons, which means $|\mathcal{L}| \geq 2$. Then by using Lemma 1 we can imply that $\mathcal{L} \leq |E| - 1$. Clearly this is a contradiction to $|\mathcal{L}| = |E|$. Therefore, we can always find a vertex v where $deg(v) = 1$ in the last step. ■

Alternatively we can use the following iterative algorithm:

Iterative Algorithm II

1. $F \leftarrow \emptyset$
2. **while** $V(G) \neq \emptyset$ **do**
3.   find a basic feasible solution of $LP_{ST}(G)$ and remove any e with $x_e = 0$.
4.   find an edge e with $x_e = 1$, then $G \leftarrow G/e, F \leftarrow F \bigcup \{e\}$.

Figure 12.3: Second Iterative Algorithm for Minimum Spanning Tree

Lemma 6 *For any bfs x with $x_e > 0$ for all edges, there is an edge e with $x_e = 1$.*

Proof. We proof this lemma by contradiction. By Lemma 4, there are $|\mathcal{L}|$ linearly independent tight constraints of the form $x(E(S)) = |S| - 1$, and $|\mathcal{L}| = |E|$. We derive a contradiction by a counting argument. Assign one token for each edge e to the smallest set in $\mathcal{L}$ that contains both endpoints of e. So there are a total of $|E|$ tokens. We show we can collect 1 token for each set and still have some extra tokens, which is a clear contradiction.

Let $S \in \mathcal{L}$ be a set have children $R_1, \ldots, R_k$ (in the laminar family). Since these are all tight sets, we have:

$$\begin{aligned} x(E(S)) &= |S| - 1 \\ x(E(R_i)) &= |R_i| - 1 \quad \text{for all } 1 \leq i \leq k \end{aligned}$$

Subtracting the sides we get:

$$x(E(S)) - \sum_i x(E(R_i)) = |S| - \sum_i |R_i| + k - 1$$

Let $A = E(S) \setminus \bigcup_i E(R_i)$. Then $x(E(A)) = |S| - \sum_i |R_i| + k - 1$. Set S gets exactly one token for each edge in A. If $A = \emptyset$ then $\chi(E(S)) = \sum_i \chi(E(R_i))$ which contradicts linear independence of $\mathcal{L}$. Also we cannot have $|A| = 1$ since $x(E(A))$ is an integer (as the right-hand side is sum/difference of sizes of a number of sets) that is positive and each x_e is assumed to be fractional. Therefore, $|A| \geq 2$, so S gets at least two tokens! ■

Using Lemma 5 or 6 it is easy to show that the iterative algorithms find a MST.

Theorem 2 *The iterative algorithms I and II find a MST.*

Proof. It only remains to show that the result is a spanning tree and this is done by induction on the number of iterations. Consider the first form of the algorithm. If we find a vertex of degree 1, say $deg(v) = 1$ then the edge incident to it must have $x_e = 1$ (since $x(\delta(v)) \geq 1$ is a constraint). Thus each edge added to F in either form of the algorithm has value 1. When e is added to F and v is removed from G note that for any spanning tree T' of $G - \{v\}$, we can build a spanning tree T of G by defining $T = T' \cup \{e\}$. So it is sufficient that we find a spanning tree in $G' = G - \{v\}$. Note that the restriction of x to $E(G')$, call it x_{res}, is a feasible solution to the LP for G'. So by induction, we find a tree F' for G' of cost at most optimum value of the LP for G'. Thus $c(F') \leq c \cdot x_res$ and $C(F) = C(F') + c_e$. Thus

$$c(F) \leq c \cdot x_{res} + c_e = c \cdot x$$

since $x_e = 1$. ■

Theorem 3 *Every basic feasible solution to this LP is integral.*

Proof. Take any basic feasible solution, and remove all the edges with $x_e = 0$. Then by using the previous lemma, we know that there are at most $|V| - 1$ linearly independent tight constraints. It implies that there are at most $|V| - 1$ non-zero variables, which means all of them are 1s. ■

12.2 Minimum Cost Bounded Degree Spanning Tree

In this section we show how iterative algorithms can solve even more general problems. Here we consider the problem of bounded degree spanning trees. Given a graph $G = (V, E)$ and a bound k, suppose we want to find a spanning tree with maximum degree at most k. This is NP-complete since with $k = 2$, it is the Hamiltonian path problem.

Theorem 4 (Furer & Raghavarchi '90) *There is a polynomial time algorithm that finds a spanning tree of maximum degree a most $k + 1$ (if there is one of with maximum degree at most k).*

As a more general case, suppose each edge of the graph has some given cost c_e. Also, each vertex v has a given bound B_v and our goal is to find a minimum cost spanning tree with degree bounded by B_v's.

Theorem 5 (Singh & Lan '07) *There is a polynomial time algorithm that finds a spanning tree of cost at most opt and degree in which every vertex v has degree at most $B_v + 1$.*

Our goal is to prove this theorem. For that end, we first formulate the problem as an integer program and consider the LP relaxation. The following LP is the relaxation for an even more general form of the problem in which we have degree bounds B_v for a subset $W \subseteq V$ of vertices. We call the following LP, LP_{BDMST}.

$$
\begin{array}{lrcll}
\min & \sum_{e \in E} c_e \cdot x_e & & & \\
\text{s.t.} & x(E(v)) & = & |V|-1 & \\
 & x(E(S)) & \leq & |S|-1 & \forall S \subset V, |S| \geq 2 \\
 & x(\delta(v)) & \leq & B_v & \forall v \in W
\end{array}
$$

For ease of exposition, we first prove a weaker version of the above theorem. We show that the following algorithm finds a tree whose cost is at most optimum and degree of every vertex v is bounded by at most $B_v + 2$.

12.2.1 Additive +2 approximation algorithm

Algorithm I for MCBDST

Input: Graph $G = (V, E)$
Output: A minimum cost bounded degree spanning tree F
1. $F \leftarrow \emptyset$
2. $B'_v = B_v$
3. **while** $V(G) \neq \emptyset$ **do**
4. find a basic feasible solution to the LP and remove any e with $x_e = 0$.
5. if there is a $v \in V$ with at most one edge $e = uv$ incident to v, then
6. $F \leftarrow F \bigcup e, G \leftarrow G - v, W \leftarrow W - v, B'_u \leftarrow B'_u - 1$
7. if there is a $v \in W$ with $deg(v) = 3$, then
8. $W \leftarrow W - v$
9. return F

Figure 12.4: First Algorithm for Minimum Cost Bounded Degree Spanning Tree

First we prove a weaker version of the main theorem mentioned above. We introduce some lemmas first and then prove that $deg \leq B_v + 2$. Consider the algorithm of Figure 12.4. At each iteration, if there is an edge e that is the only edge incident to v, and we show that we must have $x_e \geq 1$ and we pick this edge. Thus the cost we pay for an edge is not more than what the optimum pays. We argue that if there is no such vertex v with an edge of value 1 then there is a vertex $v \in W$ with $d(v) \leq 3$; so at each iteration we make progress in one of the two steps. Note that if there is a vertex v with $d(v) \leq 3$ and we remove this constraint since $B_v \geq 1$ in the worst case we will have picked all the at most 3 edges incident with v in our final solution, so the degree bound will be at most 3 which is at most $B_v + 2$.

Let $\mathcal{F} = \{S \subseteq V : x(E(S)) = |S| - 1\}$ be the set of tight set constraints. Then the following lemma can be proved similar to lemma 4 by applying uncrossing to sets in $\mathcal{F}$:

Lemma 7 *Let x be a bfs of LP_{BDMSP} with $x_e > 0$ for all edges. There is a $T \subseteq W$ with $x(\delta(v)) = B_v$ for each $v \in T$, and a laminar family $\mathcal{L} \subseteq \mathcal{F}$ such that*

1. *vectors $\{\chi(E(S)) : S \in \mathcal{L}\} \cup \{\chi(\delta(v)) : v \in T\}$ are linearly independent*
2. *vector space of $span(\mathcal{L}) \cup \{\chi(\delta(v)) : v \in T\} = span(\mathcal{F})$*
3. *$|\mathcal{L}| + |T| = |E|$*

By the argument given earlier, it is thus sufficient to prove the following lemma:

Lemma 8 *Let x be a basic feasible solution for LP with $x_e > 0$, then there is a vertex v with $deg(v) = 1$ or $v \in W$ with $deg(v) \leq 3$.*

Proof. We prove this lemma by contradiction. Suppose the lemma is not true. Then each vertex is incident to at least 2 edges, and $v \in W, deg(v) \geq 4$. Then we will have $|E| \geq \frac{2(n-|W|)+4|W|}{2} = |V| + |W|$. By using the previous lemma we have $|\mathcal{L}| + |T| = |E| \geq |V| + |W|$. Since we know that $\mathcal{L}$ doesn't have singletons, therefore $|\mathcal{L}| \leq |V| - 1$ and $|T| \leq W$. We add the two equations together and obtain $|\mathcal{L}| + |T| \leq |W| + |V| - 1$, which is clearly a contradiction to $|\mathcal{L}| + |T| = |E| \geq |V| + |W|$. ■

Theorem 6 *Algorithm I for MCBDST returns a spanning tree F of minimum cost such that the degree of $v \in F$ is at most $B_v + 2$ for $v \in W$.*

Proof. If there is a node v with $degree(v) = 1$ then since $x(\delta(v)) \geq 1$ is a valid constraint (obtained by subtracting $x(E(V - v)) \leq |V| - 2$ from $x(E(V)) = |V| - 1$), we must have $x_e \geq 1$. So we pay no more than what the LP pays at each step we pick an edge. Also, the remaining variables define a feasible solution for the residual LP, so inductively, the cost of T is at most the cost of the LP solution. As for the degree bounds, let B'_v be the current residual degree bound for a vertex v. It is easy to see that since we always pick full edges and update the degree bounds, if $v \in W$ then $deg_F(v) + B'_v = B_v$. Now when v is removed from W (because it has $deg(v) \leq 3$) then $deg_T(v) \leq deg_F(v) + 3 \leq B_v - B'_v + 3 \leq B_v + 2$, since $B'_v \geq 1$. ■

12.2.2 Additive +1 approximation algorithm

In this Section we prove Theorem 5. We start from a bfs and show that at each iteration we can either find an edge e with $x_e = 1$ (and so pick it) or there is a vertex $v \in W$ with $deg(v) \leq B_v + 1$ and we relax the constraint. The following equivalent algorithm is easier to analyze. We start from a bfs x with $x_e > 0$, for all $e \in E$. We iteratively find a vertex $v \in W$ with $deg(v) \leq B_v + 1$ and remove v from W. At the end we have the LP without any degree constraints, so it is the same LP as for MST and is thus integral.

Algorithm II for MCBDST

Input: Graph $G = (V, E)$
Output: A minimum cost bounded degree spanning tree F
1. $F \leftarrow \emptyset$
2. $B'_v = B_v$
3. **while** $V(G) \neq \emptyset$ **do**
4. find a basic feasible solution to the LP and remove any e with $x_e = 0$.
5. Let $v \in W$ be a node with $deg(v) \leq B_v + 1$
6. $W \leftarrow W - v$

Figure 12.5: Second Algorithm for Minimum Cost Bounded Degree Spanning Tree

It is easy to see that if at each iteration we find a vertex $v \in W$ with $deg(v) \leq B_v + 1$ then the degree of v at the final solution is no more than $B_v + 1$ once we remove that constraint from the LP. Also, x as it is, is feasible for the more relaxed LP. Therefore, the value of the solution for the residual (relaxed) LP is no more than optimum of LP. This implies that at the end we have a tree with cost at most optimum and degree bounds are violated

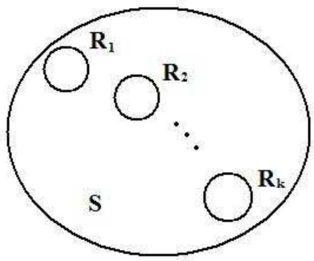

Figure 12.6: Illustration for the proof

by no more than +1. Thus we only have to show that at each iteration of the algorithm we can find such a vertex $v \in W$ to remove from W. Note that from Lemma 7 we can find the laminar family $\mathcal{L} \subseteq \mathcal{F}$ and tight degree nodes $T \subseteq W$ such that $|\mathcal{L}| + |T| = |E|$ and the corresponding constraints are linearly independent. So it is enough to prove the following theorem which will imply Theorem 5.

Theorem 7 *Let x be a basic feasible solution with $x_e > 0$ for all e, and $\mathcal{L}$ and T be the ones defined in Lemma 7. Then if $T \neq \emptyset$, there is some vertex v such that $deg(v) \leq B_v + 1$.*

Proof. We proof this theorem by contradiction.

Suppose the theorem is not true. Then we have $T \neq \emptyset$ and $deg(v) \geq B_v + 2$ for all $v \in W$.

We first show the $\forall e \in E$ with $x_e = 1$, $\chi(e) \in span(\mathcal{L})$. Let $E_1 = \{e | x_e = 1\}$. We can see that E_1 is a forest. Let $c_1, ..., c_j$ be the connected components of E_1. It is clear that each c_i is a tree. Take any component, say C and consider an ordering of its edges $\overbrace{e_1, e_2, e_3, ..., e_j}^{\text{edges of C}}, e_r$ such that the graph induced by edges $e_1, \ldots, e_j$, i.e. $C_j = \{e_1, \ldots, e_j\}$ is connected. Note that C_j is a tight set too and that $C_1, C_2, ..., C_r$ forms a laminar family of tight sets; also $\chi(e_i) = \chi(C_i) - \chi(C_{i-1})$. Therefore, $\chi(e_i) \in span(\{\chi(C_i) : 1 \leq i \leq r\})$. Take the union of these and expand it to obtain a laminar family $\mathcal{L}$ as in the proof of Lemma 7.

Now we use the token argument that is been used frequently in the proof of previous lemmas. We assign one token to each edge, which means the total number of tokens is the number of edges which is $|E|$. Then we re-distribute the tokens. The rule for the re-distribution can be described as follows. Each e gives $\frac{1-x_e}{2}$ to each of its two end points for the degree constraints and x_e tokens to the smallest set S in $\mathcal{L}$ containing both end points of e. We would like to show that each set $S \in \mathcal{L}$ and $v \in T$ gets one token and there are still some tokens left, which derives a contradiction because the total number of tokens is $|E|$.

Let's first consider vertices $v \in T$. Each such v gets $\frac{1-x_e}{2}$ for each edge of v. Then:

$$\sum_{e \in \delta(v)} \frac{1 - x_e}{2} = \frac{d(v) - \sum_{e \in \delta(v)} x_e}{2} \geq 1$$

In the above equation, $e \in \delta(v)$ means the edge incident to v. $\sum_{e \in \delta(v)} x_e = B_v$ because T has a tight degree constraint. The inequality in the equation is inferred by $d(v) \geq B_v + 2$ by our assumption.

Now consider any $S \in \mathcal{L}$. S gets x_e token for each e such that S is the smallest set containing e. Consider the case shown in Figure 12.6. We assume that S has children $R_1, \ldots, R_k$ in the laminar family. We can see that $R_1, ..., R_k$ and S are all tight sets. Hence:

$$x(E(S)) = |S| - 1$$

$$x(E(R_i)) = |R_i| - 1, i = 1, ..., k$$

Subtracting the equations of R_i's from that of S:

$$x(E(S)) - \sum_{i=1}^{k} x(E(R_i)) = |S| - 1 - \sum_{i=1}^{k}(R_i - 1).$$

Note that this is actually the number of token that S would get. Then similar to the proof of Lemma 6, by using the linear independency, we can show that the number is non-zero and integer. Therefore, S gets one token as well.

Now what is left is to show that there are still some tokens left unaccounted for. First of all, $V \in \mathcal{L}$. Otherwise the largest set is some set $S \neq V$, with $\delta(S) \neq \emptyset$, which means there exists some edge e such that $e \notin E(S)$, so there will be x_e tokens not assigned anywhere which is a contradiction. Also, if there is $v \in W \backslash T$, then by the same argument as for the case of $v \in T$ and noting that $deg(v) \geq B_v + 2$, v collects one token, which is unaccounted for. Therefore, we can never have $v \in W \backslash T$. Suppose there is $v \in V \backslash T$, then each edge e at v must have $x_e = 1$. Otherwise $\frac{1-x_e}{2} > 0$ tokens are extra, which is a contradiction. We have shown that for every such e, $\chi(e) \in span(\mathcal{L})$. And we will use this argument here. We have:

$$2\chi(E(V)) = \sum_{v \in V} \chi(\delta(v)) = \sum_{v \in T} \chi(\delta(v)) + \sum_{v \in V \backslash T} \chi(\delta(v)).$$

We showed that $V \in \mathcal{L}$ and for each $e \in \delta(v)$ with $v \in V \backslash T$, $x(e) = 1$ and so $e \in span(\mathcal{L})$. Since we assumed $T \neq \emptyset$, the constraints in T and $\mathcal{L}$ have linear dependecem, which is a contradiction.

■