Computing Science (CMPUT) 657

Algorithms for Combinatorial Games

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How to play a Sum Game?

- Review: global alphabeta can be too slow
- Incentives are great if they work
  (e.g. decomposition search, later)
- Using incentives fails when:
  - incentives are incomparable
  - subgames are too complex to compute incentives
- What else can we do?
  Use temperature/thermographs

Why Use Temperature/Thermographs?

- It is much easier to compute thermographs (TG) than canonical forms
- Memory per ~~game~~ position / game state is constant in practice
  Most TG have very few lines, 10 or more is rare
- Even knowing approximate temperatures can be a big help
  Need an algorithm to compute them

Simple Sum Strategies - Hotstrat

- Simple Sum Strategies
  - Use local data such as thermograph only
  - No global search
- Simplest idea: HotStrat
- Play in hottest subgame
  ties broken arbitrarily
- Which move? See later

HotStrat Example

- Sum game $G + H + K$
- $G = 10|8||2$
- $H = 5| - 3$
- $K = 5|1|| - 1| - 8$
- Temperatures
  - $t(G) = 7/2$
  - $t(H) = 4$
  - $t(K) = 15/4$
- H is hottest
  - HotStrat will play in H
  - for either Left or Right going first

HotStrat Example

G= 10|8||2
H= 5|–3
K= 5|1||–1|–8

t(G) = 7/2
t(H) = 4, hottest
t(K) = 15/4

Sample play of sum $G + H + K$, assuming Right goes first:

- Move $H \longrightarrow H^R = -3$
- $t(H^R) = -1$ since it is integer
- Now sum game is $G - 3 + K$, with $t(G) = 7/2, t(K) = 15/4$
- Left moves in K which is is hottest
- $K \longrightarrow K^L = 5|1$
- $t(K^L) = 2$
- Sum $G - 3 + K^L$, G is hottest, Right moves from G to 2
- Sum $2 - 3 + K^L = -1 + K^L$, Left moves K^L to $K^{LL} = 5$
- Sum $-1 + 5 = 4$. Stop reached.

HotStrat Move Selection

- Remember the construction of scaffold
- Each Left scaffold is the max of the taxed left options
- In general, at different t, different moves can ~~contribute~~

be ~~to~~ the max LS or RS at that t

- HotStrat chooses an *orthodox* move that defines the scaffold of the thermograph at the *temperature* of the game
- Ties broken arbitrarily
  Tie: different options have the same value at $t(G)$)

Definition - Orthodox Move

- $L_t(G)$ = Left scaffold at temperature t
- $R_t(G^L)$ = Right scaffold of option G^L at t
- Move to G^L is orthodox at temperature t (or t-orthodox) iff taxed right option equals left scaffold of G at t
- $R_t(G^L) - t = L_t(G)$
- Likewise for right options (tax: $+t$):
- G^R is orthodox iff $L_t(G^R) + t = R_t(G)$

Weakness of HotStrat

- We have seen before that the hottest move is not always optimal
- How bad can it get?
- We would like an error bounded by the temperature of the hottest subgame
- Remember $t(G + H) \leq max(t(G), t(H))$
- Unfortunately, HotStrat can lose much more
- It can be tricked into unbounded loss
  Loss grows without a limit, as the number of subgames grows
- Note: this is worst-case performance. Not too bad in practice

Hotstrat is

See empirical results in the "Locally informed.." paper

Weakness of HotStrat - Example

- $G = \{10||0| - 20\}| - 4$, $t(G) = 2$
- $H = 10||9| - 9$, $t(H) = 1$
- $m(G) = -2, m(H) = 9$, so $m(G + H) = 7$
- Play $G + H$, Left goes first and follows HotStrat:
- G hottest, so $G \longrightarrow G^L = \{10||0| - 20\}$, $t(G^L) = 10$
- Right *does not follow HotStrat*, plays $H \longrightarrow H^R = \{9| - 9\}$, $t(H^R) = 9$
- G^L is hottest, so Left plays there to 10, then Right plays H^R to -9
- Total score 10 - 9 = 1
- Both played the same number of moves

Weakness of HotStrat - Watching the Means

- $G = \{10||0| - 20\}| - 4$, $m(G) = -2$
- $H = 10||9| - 9$, $m(H) = 9$
- Left plays $G \longrightarrow G^L = \{10||0| - 20\}$,
- $m(G^L) = 0$, Left gains 2 points over $m(G) = -2$
- Right $H \longrightarrow H^R = \{9| - 9\}$
- $m(H^R) = 0$, Right gains 9 points over $m(H) = 9$
- Left $G^L \longrightarrow 10$, Left gains 10 points
- Right plays H^R to -9, Right gains 9 points
- Total gains Left - Right = (2+10) - (9+9) = -6 for Left
- Observe how means of subgames shifted: Right gained 9 two times, Left only got 2+10

Discussion

- $m(G+H) = 7$, $t(G+H) \leq 2$
- Left goes first, can expect to get to around m+t = 9
- However, Left only got to a stop of 1
- Left even lost 6 points compared to the mean, far more than the temperature
- And Left even went first!
- If we play n copies of $G+H$:
- We should expect to get around n times $m(G+H) = 7n$
- With the strategies above, all n copies will play out the same way.
- Left only gets a score of 1 per copy
- Loses about $6n$ points, grows to infinity with increasing n
- Unbounded loss

How to Fix HotStrat? - Watching the Means

- $G = \{10||0|-20\}|-4$, $m(G) = -2$
- $H = 10||9|-9$, $m(H) = 9$
- Left plays $G \longrightarrow G^L = \{10||0|-20\}$,
- $m(G^L) = 0$, Left gains 2 points
- Right $H \longrightarrow H^R = \{9|-9\}$
- $m(H^R) = 0$, Right gains 9 points
- **Left answers** $H^R \longrightarrow H^{RL} = 9$
- $m(H^R L) = 9$, Left gains 9 points
- Right $G^L \longrightarrow G^{LR} = \{0|-20\}$
- $m(G^{LR}) = -10$, Right gains 10 points
- Left $G^{LR} \longrightarrow 0$
- $m(0) = 0$, Left gains 10 points

How to Fix HotStrat? - Watching the Means

- Total gains Left - Right = (2+9+10) - (9+10) = +2 for Left
- Compare with previous result.
- Left "defended" the mean of H, end result in H is $H^{RL} = 9$.
- Left gained 2 points from going first in G, $m(G) = -2$ and end result in G is $G^{LRL} = 0$

How to Fix HotStrat?

- In the example, Left should answer Right's *temperature-increasing* move. $H^R \longrightarrow H^{RL} = 9$
- Then Right moves $G^L = \{10||0| - 20\}$ to $G^{LR} = \{0| - 20\}$
- Finally, Left moves to G^{LR} to 0, total score +9
- Compare with previous result. Left "defended" the mean of H.

Refinements of HotStrat

- HotStrat+ (Müller and Li 2004)
  - play sente moves earlier
  - A little better in practice
  - no bounded loss either
- ThermoStrat (Winning Ways)
  - **bounded loss** (by temperature of second hottest game)
  - pretty complex
- **SenteStrat** (Berlekamp 1996)
  - Simpler than ThermoStrat
  - Also has **bounded loss**
  - Focus on this method here

Simple Sum Strategy - SenteStrat

- SenteStrat (Berlekamp 1996):
  - Answer opponent *sente* moves in the same subgame
  - Play HotStrat otherwise
- How is sente defined here?
- Answer: relative to an *ambient* a:
  the lowest temperature reached so far
- An opponent move is considered *sente* if:
  - They move any subgame G_i to a temperature $t > a$
  - In all such cases, SenteStrat replies in the same G_i
- SenteStrat is a simple algorithm with bounded loss

SenteStrat on Our Example

- $G = \{10||0| - 20\}| - 4$, $t(G) = 2$
- $H = 10||9| - 9$, $t(H) = 1$
- Ambient a =max$(t(G), t(H)) = 2$
- Left plays HotStrat, $G \longrightarrow G^L = 10||0| - 20$
- Again Right tries $H \longrightarrow H^R = 9| - 9$
- $t(H^R) = 9 > a$
- Left treats this move as sente and answers in H^R
- $H^R \longrightarrow H^{RL} = 9$
- Play then continues the same as in good sequence above

Summary and Outlook

- Algorithms for sums of “relatively simple” subgames
- We can compute means, temperatures, thermographs of subgames
- Discussed simple “no-search” algorithms: HotStrat, SenteStrat
- Adding a little bit of search helps in practice, see (Müller and Li 2004) paper