Computing Science (CMPUT) 657

Algorithms for Combinatorial Games

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Part III

Algorithms for Combinatorial Games

The Economist's View of Combinatorial Games

- Elwyn Berlekamp, Games of No Chance 1996
- Seminal paper
- Importance for us:
  - Computational/algorithmic implications
- 41 pages, many results and many topics
- We read selected parts
- We already know the content of some sections
- We will not discuss ko-related things

Some Contributions

- Introduces subzero thermography
- Two new methodologies for defining and computing a position's mean and temperature
- Competitive auctions to bid for the right to play
- Economic rules
- Coupon stacks and enriched environments
- Extended thermography and applications to loopy games in Go
    - We skip this part

Two main questions of CGT

- Two traditional main questions
- Who is ahead, and by how much?
  - Exact "classical" answer: value (in canonical form)
  - Rough "classical" answer: mean
- How big is the next move?
  - Exact "classical" answer: incentive
  - Rough "classical" answer: temperature
- New with economic rules:
  - Mean and temperature **are exact** answers
  - How can it be?
  - We're playing a (slightly) different game

Summary of Economic Rules for Playing a Combinatorial Game

- Step 1: Choose who plays which color
- Step 2: Bid for two things:
  - Current tax rate
  - The right to play first
- Each move: play and pay the tax, or pass
- Both pass: bid again for a lower tax rate
- Both pass at minimum tax: Game over
- Details on next slides

Choosing Colors

- Both players bid for the right to play Black
  - one player bids $b1$
  - other player bids $b2$
  - assume $b1 \geq b2$
- Compute mean $m = (b1 + b2)/2$, so $b1 \geq m \geq b2$
- Higher bidder gets black
- Pays m to opponent
- Both should be happy now (discuss why)
- Question: How much should one bid?
- Answer: see later

Playing the Game

- Another bid, for the current tax you're willing to pay
- Higher bid t wins and plays first at that tax
- For each followup move each player has a choice
  - Pay the tax t to the opponent and play a move in game
  - Pass
- Play at tax rate t continues until both players pass in turn

Example - Choosing Colors

- Example:
- Bid for playing Black:
  - Students bid 3
  - I bid 2
- Higher bid: students
- Average bid: 2.5
- Students play black, pay me 2.5 for the privilege
- Both sides are happy...

Example - Starting the Game

- Bid for tax rate/going first:
  - Students bid 4, I bid 4.5
  - Tax rate = 4.5, I go first
- Your turn, two choices:
  - 1. you play and pay 4.5$ to me
  - 2. you pass (at this tax rate)
- Example: I pay 4.5 and play,
- You pass, then I pass as well
- Now we bid again for a new (lower) tax rate

Lowering the Tax

- Both players passed in turn
- Next step: re-bid for lower tax
- Again, higher bid “wins”, plays first
- Continue play as before, but at the lower tax rate...
- ...until both pass again
- Repeat this loop until...

End of Game

- ... both pass at the *minimum tax rate*
- There is a pre-defined minimum tax rate
  - 0 or -1 makes the most sense. We'll use -1
- The game is over
- Who wins?
- The player with *more money* wins
- **Not** the player who made the last move!
- Let's play the game in the handout. Two teams

Handout

Play the sum game G under Economic Rules

- G = G1 + G2 + G3 + G4 + G5
- G1 = 2|0
- G2 = 10|1 || -1|-10
- G3 = 20|5 || 0
- G4 = -3 | 5
- G5 = 8|7 || 6|5 ||| 0|-1 || -2|-3

The Book Strategy

- Inspiration: Paul Erdos' book of all "perfect" mathematics
- "Contains all mathematical truths in their most revealing forms"
- Our more modest book:
  - Shows the mean and temperature of all positions in all subgames
- Berlekamp showed that this information is enough to **play perfectly under Economic Rules**

Book Strategy

- Initially, bid the mean of the sum for the privilege of playing Left
- Then play a kind of Sentestrat
- Remember ambient = max. temperature of sum
- There is a current tax rate,
  not necessarily the same as the ambient

Sentestrat-like Book Strategy

- Sentestrat-like strategy:
- Reply if opponent's move left a position hotter than current tax rate
- Otherwise, if there is a game hotter than current tax rate, play hottest subgame

  hotter-or-equal-to
- Otherwise pass. If opponent also passes:
- Bid temperature of hottest subgame, if you have a "canonical move" left

Facts that Justify Book Strategy

- The value of playing Left in any subgame under Economic Rules is equal to the mean of that subgame
- It is always safe (no loss) to pay the temperature of a subgame as a tax
  - (Sometimes we could pay more. See ~~upcoming~~ discussion of Hotstrat+)
- As Example, let's work out the Book Strategy for the Handout Game

Book Strategy Guarantees the Minimax Value

- Theorem: Book Play Ensures No Monetary Loss
- Let μ be the mean of a sum game
- Play under Economic Rules
- If Left plays Book Strategy, gets at least μ dollars
- If Right plays Book Strategy, Left gets at most μ dollars
- Note: of course this is without the initial bid for the right to play black. Both players would bid exactly μ dollars if they are competent, and the result would be zero.
- So μ is the minimax value of the game, and both players can follow the book strategy to achieve that value against any opponent.

Stable, Unstable, and Semistable Positions

- Next important new concepts in paper:
- Stable, unstable, and semistable positions
- First some notation
- Our discussion is a bit more detailed than in the paper

Ancestors and Positions of a Game

- Given a game G
- A game H is called “a position of G” if it occurs somewhere in the subtree of G
- H is option of G, or option of option of G, or ...
- Given G and position H of G, there is a path $P_0, P_1, ..., P_n$ of ancestors of H in G
- $P_0 = G$,
  P_1 is some option of P_0
  ...
  H is some option of P_n

Stable, Unstable, and Semistable Positions

- Definition: Position H of a game G is *stable in G* if all ancestors of H in G have temperatures greater than H's.

$$t(H) < t(P_i) \text{ for all } i$$

- Definition: H is *unstable* or *transient* in G if it has an ancestor of lower temperature.

$$\text{There exists } i : t(H) > t(P_i)$$

- Definition: If H is not unstable, but has one or more ancestors of temperature equal to its own, it is called *semistable*.

$$\text{not exists } i : t(H) > t(P_i) \text{ and exists } i : t(H) = t(P_i)$$

- Compare with concept of ambient from Sentestrat

Quote from Berlekamp

Semi-stable positions can be treated as stable or as unstable. Treating them as stable yields the simplest and most convenient economic analyses. Treating them as unstable leads to more refined, more complicated algorithms that exploit the calculus of infinitesimal games to get the last move at each t, whenever possible.

- Comment from Martin: I have never seen such an algorithm written down...
- This would be a good course project...

Small Comment on Sentestrat and the Quote Above

- SenteStrat keeps playing locally until a stable position is reached
- When exactly to stop? $t \geq$ ambient or $t >$ ambient?
- In regular play (not Economic rules) we gain something from making the last move at a temperature

Mainlines and Sidelines, Sente and Gote

- Remember the original definition of mean as the limit of $LS(nG)/n$ as n goes to infinity
- Lines of play that contribute to the mean computation are called *mainlines*
- They will be played infinitely often as n goes to infinity
- Other lines are called *sidelines*
- They are only played a few times, or not at all

Example - Gote

- Example: gote game $G = 2|0$
- Play 1000 copies of G
- Half of the copies (500) will end up in 2, the other half in 0
- Both lines of play are *mainlines*

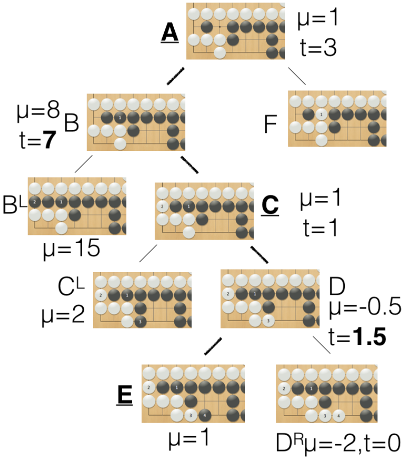
Example - Sente

- Example: sente game $H = 20|5||0$
- Play 1000 copies of H
- At least 999 copies will end up in 5
- At most one will end up in 0
- Only the line of play $H \rightarrow 20|5 \rightarrow 5$ is a *mainline*
- The lines leading to 0 or 20 are *sidelines*

Second Sente Example and Thermographs

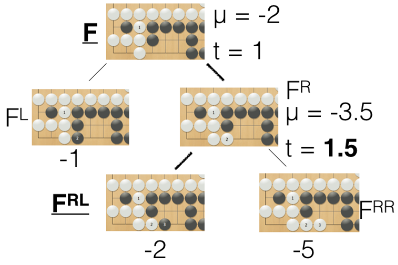
- Sente game for Left, 10|2 || -2
- Sequence leading to 2 is the only mainline
- Sequences to 10, -2 are sidelines
- Insight: Thermographs of sideline positions have *no effect* on the means of any of their ancestors
- Sidelines often do affect the temperature(s) of some of their near ancestors

Go Example from Paper



- Mainline A,B,C,D,E
- Only A,C,E are stable on mainline
- In B,D temperature is higher than in parent - transient position
- The mean comes only from the mainline(s)
- We need to understand *some* information about many sidelines to determine what the mainline is...

Position F Details

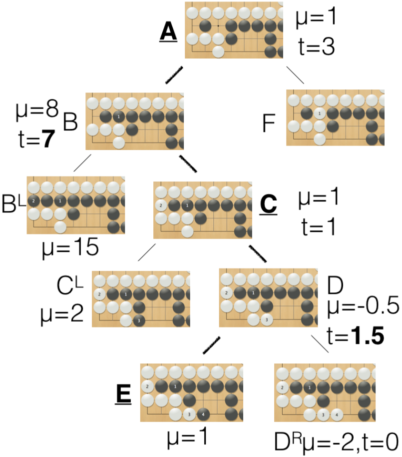


$F = \{-1 \mid \{-2 \mid -5\}\}$

$F^R = \{-2 \mid -5\}$

- This thick line is a mainline from F
- But F itself was not on a mainline from A (previous slide)

More on Go Example from Paper



- In example, only one mainline
- Play does *not* need to alternate along a mainline
- E.g. in example move 1 is sente for B. After sequence 1-2, C is stable follower
- C is a sente position for White. White can play sequence 3-4 in sente, but later

Rich Environment

- Intuitively, in a rich environment there are lots of moves at all temperatures
- You cannot gain a lot by “getting the last big move”
- The situation is much simplified
- Basically, only mainlines matter in a rich environment
- “Sidelines will not happen”

Section 2.4 Top-down Thermography

- Fragments of an algorithm?
- What do we need to turn these comments into a full algorithm?
- Goal: “alphabeta for means and temperatures”
- Kao’s two papers (in optional readings) are a first step in this direction

Keep Reading The Economist's View...

- 2.5 Subzero Thermography we discussed already
- 3.3 Sentestrat, Thermostrat, and Hotstrat also done
- Section 4: Sentestrat and Thermostrat both ensure perfect play according to economic rules.
  In "classical" rules, they ensure at least the mean for the first player
- In "rich environment", first player can expect about $t/2$ more where t is highest temperature
- Note: in "enriched environment", this will be exactly true
- 4.3 is **new and important**: enriched environments. We will look at "coupon stacks" soon

Summary

- Different, "economic" rules
- Greatly simplifies playing sums of hot games
- Knowing means and temperatures is enough for perfect play here
- Preview: we can turn it around and *play* such games in order to compute means and temperatures